Tutorial on Analytic Algorithms to Solve Cubic and Quartic Equations

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This tutorial works out solutions to three cubic equations and three quartic equations by using algorithms that are fully described in the companion papers. First the three cubic equations are solved. Then the Ferrari modified algorithm solves the first quartic equation, and the National Bureau of Standards (NBS) modified algorithm solves the second. The Euler modified and Ferrari modified algorithms are each used to solve the third quartic equation. Appendix A plots example cubic and quartic polynomials to show how the number of real roots is related to the shape of the functional curve. Appendix B provides a review of the mathematics needed to use and derive the algorithms.

Solving the Cubic Equation

The practical algorithm for solving the cubic equation is tabulated below. Following the algorithm are equations to check the calculated solutions. The algorithm inputs are three real coefficients a_2, a_1, and a_0, and the outputs are the three values z_1, z_2, and z_3 such that

$$p(z) \;=\; z^3 + a_2\, z^2 + a_1\, z + a_0 \;=\; (z - z_1)\,(z - z_2)\,(z - z_3) \quad \text{for all } z.$$

The outputs are thus the three solutions of the general cubic equation

$$z_n^3 + a_2\, z_n^2 + a_1\, z_n + a_0 = 0, \qquad n = 1, 2, 3.$$

Solution z_1 is defined as the greatest real solution. Solutions $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$ are either real numbers ($y_2 = y_3 = 0$) or a complex conjugate pair ($x_2 = x_3$, $y_2 = -y_3$). All angle values are in radian measure.

FIGURE 1 PRACTICAL ALGORITHM FOR SOLVING THE CUBIC EQUATION

<u>Given</u>: Real coefficients a_2, a_1, and a_0,	
<u>Find</u>: z_1, $z_2=x_2+iy_2$, and $z_3=x_3+iy_3$ such that $z^3 + a_2\, z^2 + a_1\, z + a_0 = (z - z_1)(z - z_2)(z - z_3)$ for all z.	
<u>Calculate q and r</u>: $q = \frac{a_1}{3} - \frac{a_2^2}{9}$ $\quad r = \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27}$	
Case: 1: $r^2 + q^3 > 0$ ⇔ Only One Real Solution (*Numerical Recipes*)	**Case 2: $r^2 + q^3 \le 0$ ⇔ Three Real Solutions (Viète)**
$A = \left(\lvert r\rvert + \sqrt{r^2 + q^3}\right)^{1/3}$	$\theta = \begin{cases} 0 & \text{if } q = 0 \\ \mathrm{Cos}^{-1}[r/(-q)^{3/2}] & \text{if } q < 0 \end{cases} \qquad 0 \le \theta \le \pi$
$t_1 = \begin{cases} A - q/A & \text{if } r \ge 0 \\ q/A - A & \text{if } r < 0 \end{cases}$	$\phi_1 = \theta/3 \qquad \phi_2 = \phi_1 - 2\pi/3 \qquad \phi_3 = \phi_1 + 2\pi/3$
$z_1 = t_1 - \frac{a_2}{3} \qquad x_2 = x_3 = -\frac{t_1}{2} - \frac{a_2}{3}$	$z_1 = 2\sqrt{-q}\,\cos\phi_1 - a_2/3$
$y_2 = -y_3 = \frac{\sqrt{3}}{2}\left(A + \frac{q}{A}\right)$	$z_2 = x_2 = 2\sqrt{-q}\,\cos\phi_2 - a_2/3 \qquad y_2 = 0$
$z_2 = x_2 + iy_2 \qquad z_3 = x_2 - iy_2$	$z_3 = x_3 = 2\sqrt{-q}\,\cos\phi_3 - a_2/3 \qquad y_3 = 0$
	$z_3 \le z_2 \le z_1$

Validate the Calculated Solutions

Validate calculated solutions z_1, $z_2 = x_2 + iy_2$, and $z_3 = x_3 + iy_3$ by reproducing the input coefficients according to the following check equations:

$$a_2 = -(z_1 + x_2 + x_3) \qquad a_1 = z_1(x_2+x_3) + x_2x_3 + y_2^2 \qquad a_0 = -z_1(x_2x_3+y_2^2).$$

Three sample problems are worked out below in order to demonstrate operation of the algorithm.

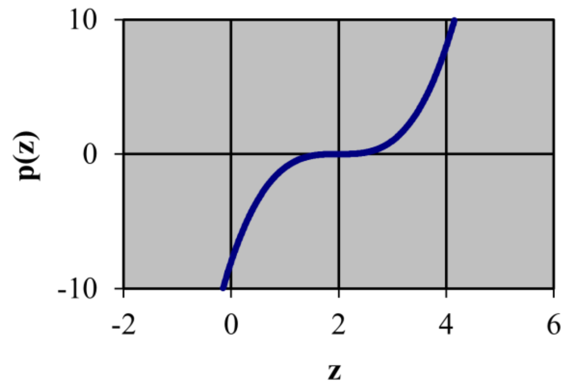
Cubic Problem 1: Three Equal Real Solutions: $r = q = 0$

Problem: Find the solutions z_1, $z_2 = x_2 + iy_2$, and $z_3 = x_3 + iy_3$ of

$$z_n^3 - 6z_n^2 + 12z_n - 8 = 0$$

Given: $a_2 = -6$, $a_1 = 12$, $a_0 = -8$

Solution:



$$q = \frac{a_1}{3} - \frac{a_2^2}{9} = \frac{12}{3} - \frac{(-6)^2}{9} = 4 - 4 = 0$$

$$r = \frac{a_1a_2 - 3a_0}{6} - \frac{a_2^3}{27} = \frac{12(-6) - 3(-8)}{6} - \frac{(-6)^3}{27} = -8 + 8 = 0$$

Because $r^2 + q^3 \le 0$, the algorithm uses Case 2, the Viète method. With $q = 0$, the values of θ and the ϕ_n are irrelevent. The solutions are

$$z_1 = z_2 = z_3 = x_2 = x_3 = -a_2 / 3 = -(-6) / 3 = 2, \qquad y_2 = y_3 = 0.$$

$$z_1 = 2, \qquad z_2 = 2, \qquad z_3 = 2$$

Check Solutions Against the Input Coefficients: $a_2 = -6$, $a_1 = 12$, $a_0 = -8$:

$a_2 = -(z_1 + x_2 + x_3) = -(2 + 2 + 2) = -6$ ✓

$a_1 = z_1(x_2+x_3) + x_2x_3 + y_2^2 = 2(2 + 2) + 2(2) + 0^2 = 12$ ✓

$a_0 = -z_1(x_2x_3+y_2^2) = -(2)[2(2) + 0^2] = -8$ ✓

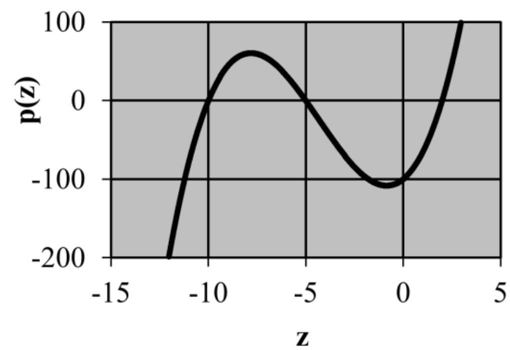
Note: The cubic equation of Cubic Problem 1 is a special case of the form

$$(z_n - \alpha)^3 = z_n^3 - 3\alpha z_n^2 + 3\alpha^2 z_n - \alpha^3 = 0 \quad \text{for any real } \alpha.$$

Thus, $a_2 = -3\alpha$, $a_1 = 3\alpha^2$, $a_0 = -\alpha^3$. Values q and r equal zero, and the three solutions are $z_1 = z_2 = z_3 = \alpha$. In Cubic Problem 1, $\alpha = 2$.

Cubic Problem 2: Three Real Solutions, Not All Equal

$r^2 + q^3 \le 0,\ \ q \ne 0$



Problem: Find the solutions z_1, $z_2 = x_2 + iy_2$, and $z_3 = x_3 + iy_3$ of

$$\boxed{z_n^3 + 13\, z_n^2 + 20\, z_n - 100\ =\ 0}$$

Given: $a_2 = 13,\ \ a_1 = 20,\ \ a_0 = -100$

Solution:

$$q = \frac{a_1}{3} - \frac{a_2^2}{9} = \frac{20}{3} - \frac{13^2}{9} = -12.11111111$$

$$r = \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27} = \frac{20(13) - 3(-100)}{6} - \frac{13^3}{27} = 11.96296296$$

$$r^2 + q^3\ =\ (11.96296296)^2 + (-12.11111111)^3\ =\ -1633.33333333$$

Because $r^2 + q^3 \le 0$, the algorithm uses Case 2, the Viète method.

$$\frac{r}{(-q)^{3/2}} = \frac{11.96296296}{[-(-12.11111111)]^{3/2}} = \frac{11.96296296}{42.14790405} = 0.28383293$$

$$\theta\ =\ \mathrm{Cos}^{-1}[r/(-q)^{3/2}]\ =\ \mathrm{Cos}^{-1}(0.28383293)\ =\ 1.28300725, \qquad 2\pi/3\ =\ 2.09439510$$

$$\phi_1 = \theta/3 \quad = 1.28300725\,/\,3 = \quad 0.42766908 \qquad \cos(\phi_1) = \ \ 0.90993497$$

$$\phi_2 = \phi_1 - 2\pi/3\ = 0.42766908 - 2.09439510 = -1.66672602 \qquad \cos(\phi_2) = -0.09578263$$

$$\phi_3 = \phi_1 + 2\pi/3\ = 0.42766908 + 2.09439510 = \quad 2.52206418 \qquad \cos(\phi_3) = -0.81415234$$

$$2\sqrt{-q}\ =\ 2\sqrt{12.11111111}\ =\ 6.96020434 \qquad a_2/3\ =\ 13\,/\,3\ =\ 4.33333333$$

$$z_1 \quad = 2\sqrt{-q}\ \cos(\phi_1) - a_2/3 = 6.96020434\ (\ 0.90993497) - (4.33333333)\ = \quad 2$$

$$z_2 = x_2 = 2\sqrt{-q}\ \cos(\phi_2) - a_2/3 = 6.96020434\ (-0.09578263) - (4.33333333)\ = \quad -5$$

$$z_3 = x_3\ = 2\sqrt{-q}\ \cos(\phi_2) - a_2/3 = 6.96020434\ (-0.81415234) - (4.33333333)\ =\ -10$$

$y_2 = y_3 = 0$

$$\boxed{z_1\ =\ 2, \qquad z_2\ =\ -5, \qquad z_3\ =\ -10}$$

Check Solutions Against the Input Coefficients: $a_2 = 13,\ a_1 = 20,\ a_0 = -100$:

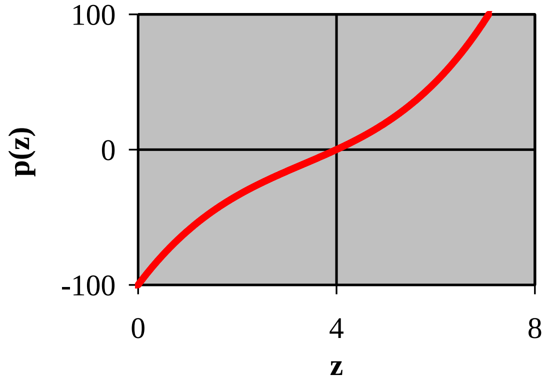
$$a_2\ =\ -(z_1 + x_2 + x_3) \qquad =\ -[2 + (-5) + (-10)] \qquad =\quad 13\ ✓$$

$$a_1\ =\ z_1(x_2 + x_3) + x_2 x_3 + y_2^2\ \ =\ 2(-5-10) + (-5)(-10) + 0^2\ =\quad 20\ ✓$$

$$a_0\ =\ -z_1(x_2 x_3 + y_2^2) \qquad =\ -2[(-5)(-10) + 0^2] \qquad =\ -100\ ✓$$

Cubic Problem 3: One Real Solution and a Complex Conjugate Pair

$r^2 + q^3 > 0$



Problem: Find roots z_1, $z_2 = x_2 + iy_2$, and $z_3 = x_3 + iy_3$ of

$$z_n^3 - 10\,z_n^2 + 49\,z_n - 100 = 0$$

Given: $a_2 = -10$, $a_1 = 49$, $a_0 = -100$

Solution:

$$q = \frac{a_1}{3} - \frac{a_2^2}{9} = \frac{49}{3} - \frac{(-10)^2}{9} = 5.22222222$$

$$r = \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27} = \frac{49(-10) - 3(-100)}{6} - \frac{(-10)^3}{27} = 5.37037037$$

$$r^2 + q^3 = (5.37037037)^2 + (5.22222222)^3 = 171.25925926$$

Because $r^2 + q^3 > 0$, the algorithm uses Case 1, the *Numerical Recipes* solution.

$$A = \left(|r| + \sqrt{r^2 + q^3}\right)^{1/3} = \left(5.37037037 + \sqrt{171.259259259}\right)^{1/3} = 2.64273441$$

$t_1 = A - q/A$ if $r \geq 0$, $t_1 = q/A - A$ if $r < 0$. We have $r \geq 0$. Therefore,

$$t_1 = A - q/A = 2.64273441 - 5.22222222 / 2.64273441 = 0.66666667$$

$$a_2 / 3 = -10 / 3 = -3.33333333$$

$$z_1 = t_1 - \frac{a_2}{3} = 0.66666667 - (-3.33333333) = 4$$

$$x_2 = x_3 = -\frac{t_1}{2} - \frac{a_2}{3} = -\frac{0.66666667}{2} - (-3.33333333) = 3$$

$$y_2 = -y_3 = \frac{\sqrt{3}}{2}\left(A + \frac{q}{A}\right) = \frac{\sqrt{3}}{2}(2.64273441 + 5.22222222 / 2.64273441) = 4$$

$$z_1 = 4, \quad z_2 = 3 + 4i, \quad z_3 = 3 - 4i$$

Check Solutions Against the Input Coefficients: $a_2 = -10$, $a_1 = 49$, $a_0 = -100$:

$a_2 = -(z_1 + x_2 + x_3) = -(4 + 3 + 3) = -10$ ✓

$a_1 = z_1(x_2+x_3) + x_2x_3 + y_2^2 = 4(3+3) + 3(3) + 4^2 = 49$ ✓

$a_0 = -z_1(x_2x_3+y_2^2) = -4[3(3) + 4^2] = -100$ ✓

Solving the Quartic Equation

Three sample problems are worked out below in order to demonstrate the operation of the Ferrari, Euler, and NBS modified algorithms tabulated below. Equations to check the calculated solutions are also tabulated. The Ferrari modified algorithm solves the first problem, and the NBS modified algorithm solves the second. The Euler modified and Ferrari modified algorithms are each used to solve the third problem.

The algorithm inputs are four real coefficients A_3, A_2, A_1, and A_0, and the outputs are the four values Z_1, Z_2, Z_3 and Z_4 such that

$$P(Z) = Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \quad \text{for all } Z.$$

The outputs are thus the four solutions of the general quartic equation

$$Z_n^4 + A_3 Z_n^3 + A_2 Z_n^2 + A_1 Z_n + A_0 = 0, \quad n = 1, 2, 3, 4.$$

Solutions $Z_1 = X_1 + iY_1$ and $Z_2 = X_2 + iY_2$ are either real numbers ($Y_1 = Y_2 = 0$) or form a complex conjugate pair ($X_1 = X_2$, $Y_1 = -Y_2 > 0$). Solutions $Z_3 = X_3 + iY_3$ and $Z_4 = X_4 + iY_4$ are either real numbers ($Y_3 = Y_4 = 0$) or form a complex conjugate pair ($X_3 = X_4$, $Y_3 = -Y_4 > 0$). All angle values are in radian measure.

Ferrari Modified Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3/4$, $b_2 = A_2 - 6C^2$,
$b_1 = A_1 - 2A_2C + 8C^3$,
$b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Solve this resolvent cubic equation for real m:

$$m^3 + b_2 m^2 + (b_2^2/4 - b_0)m - b_1^2/8 = 0.$$

Use a real solution $m > 0$ if it exists. Otherwise, $m = 0$.

$$\Sigma = \begin{cases} 1 & \text{if } b_1 > 0 \\ -1 & \text{otherwise} \end{cases}$$

$$R = \Sigma\sqrt{m^2 + b_2 m + b_2^2/4 - b_0}$$

$$Z_{1,2} = \sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 - R}$$

$$Z_{3,4} = -\sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 + R}$$

where the radicand in the formula for R is nonnegative provided that real $m > 0$ is used if it exists.

Euler Modified Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3/4$, $b_2 = A_2 - 6C^2$,
$b_1 = A_1 - 2A_2C + 8C^3$,
$b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Find the three solutions r_1, r_2, and r_3 of the resolvent cubic equation:

$$r_k^3 + (b_2/2)\, r_k^2 + [(b_2^2 - 4b_0)/16]\, r_k - b_1^2/64 = 0.$$

Solution r_1 is the greatest real solution and $r_1 \geq 0$. Solutions $r_2 = x_2 + iy_2$ and $r_3 = x_3 + iy_3$ are real ($y_2 = y_3 = 0$), or they form a complex conjugate pair ($x_2 = x_3$, $y_2 = -y_3 > 0$).

$\Sigma = 1$ if $b_1 > 0$, $\Sigma = -1$ otherwise.

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{x_2 + x_3 - 2\Sigma\sqrt{x_2x_3 + y_2^2}}$$

$$T_{3,4} = -\sqrt{r_1} \pm \sqrt{x_2 + x_3 + 2\Sigma\sqrt{x_2x_3 + y_2^2}}$$

where $x_2x_3 + y_2^2 \geq 0$.

$$Z_n = T_n - C, \quad n = 1, 2, 3, 4$$

NBS Modified Algorithm

Problem: Given real coefficients A_3, A_2, A_1, and A_0, find Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Solution: Calculate u_1 as the greatest real solution of the resolvent cubic equation

$$u^3 - A_2u^2 + (A_1A_3 - 4A_0)u + 4A_0A_2 - A_1^2 - A_0A_3^2 = 0.$$

$$\Sigma_g = \begin{cases} 1 & \text{if } A_1 - A_3u_1/2 > 0 \\ -1 & \text{otherwise} \end{cases}$$

$$p_1 = A_3/2 - \sqrt{A_3^2/4 + u_1 - A_2} \qquad p_2 = A_3/2 + \sqrt{A_3^2/4 + u_1 - A_2}$$

$$q_1 = u_1/2 + \Sigma_g\sqrt{u_1^2/4 - A_0} \qquad q_2 = u_1/2 - \Sigma_g\sqrt{u_1^2/4 - A_0}$$

$$Z_{1,2} = -p_1/2 \pm \sqrt{p_1^2/4 - q_1} \qquad Z_{3,4} = -p_2/2 \pm \sqrt{p_2^2/4 - q_2}$$

Validate Calculated Solutions

Validate calculated solutions $Z_1 = X_1 + iY_1$, $Z_2 = X_2 - iY_1$, $Z_3 = X_3 + iY_3$, and $Z_4 = X_4 - iY_3$ by reproducing the input coefficients according to the following check equations:

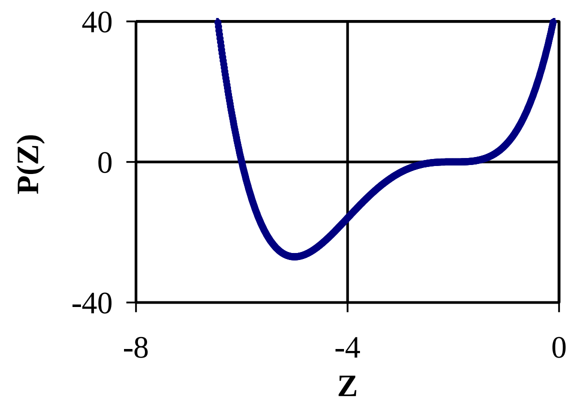
$$A_3 = -(X_1+X_2+X_3+X_4)$$

$$A_2 = X_1X_2+Y_1^2+(X_1+X_2)(X_3+X_4)+X_3X_4+Y_3^2$$

$$A_1 = -[(X_1X_2+Y_1^2)(X_3+X_4) + (X_3X_4+Y_3^2)(X_1+X_2)]$$

$$A_0 = (X_1X_2+Y_1^2)(X_3X_4+Y_3^2).$$

Quartic Problem 1: Two Distinct Real Solutions, The Greater with Multiplicity of Three



Problem: Find solutions Z_1, Z_2, Z_3, and Z_4 of

$$\boxed{Z_n^4 + 12\,Z_n^3 + 48\,Z_n^2 + 80\,Z_n + 48 = 0}$$

Given: $A_3 = 12,\ A_2 = 48,\ A_1 = 80,\ A_0 = 48$

Solution with Ferrari Modified Algorithm:

$C = A_3/4 = 12/4 = 3$

$b_2 = A_2 - 6C^2 = 48 - 6{\cdot}3^2 = -6$

$b_1 = A_1 - 2A_2C + 8C^3 = 80 - 2{\cdot}48{\cdot}3 + 8{\cdot}3^3 = 8$

$b_0 = A_0 - A_1C + A_2C^2 - 3C^4 = 48 - 80{\cdot}3 + 48{\cdot}3^2 - 3{\cdot}3^4 = -3$

Resolvent cubic equation is $m^3 + b_2\,m^2 + (b_2^2/4 - b_0)m - b_1^2/8 = 0$ with coefficients:

$b_2 = -6,\qquad b_2^2/4 - b_0 = (-6)^2/4 - (-3) = 12,\qquad -b_1^2/8 = -(8)^2/8 = -8.$

$$\boxed{m^3 - 6\,m^2 + 12m - 8 = 0:\quad \text{Real solution is } m = 2 \text{ from Cubic Problem 1.}}$$

$b_1 = 8 > 0 \Rightarrow \Sigma = 1.\qquad R = \Sigma\sqrt{m^2 + b_2 m + b_2^2/4 - b_0}$

$R = (1)\sqrt{2^2 - 6(2) + (-6)^2/4 - (-3)} = \sqrt{4 - 12 + 9 + 3} = 2$

$Z_{1,2} = \sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 - R} = \sqrt{2/2} - 3 \pm \sqrt{-2/2 - (-6)/2 - 2}$

$Z_{3,4} = -\sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 + R} = -\sqrt{2/2} - 3 \pm \sqrt{-2/2 - (-6)/2 + 2}$

$Z_{1,2} = -2 \pm \sqrt{0} = -2\qquad\qquad Z_{3,4} = -4 \pm \sqrt{4} = -4 \pm 2$

$$\boxed{Z_1 = Z_2 = Z_3 = -2,\qquad Z_4 = -6}$$

$$(X_1 = X_2 = X_3 = -2,\qquad Y_1 = Y_3 = 0)$$

Check Solution Against the Input Coefficients: $A_3 = 12,\ A_2 = 48,\ A_1 = 80,\ A_0 = 48$:

$A_3 = -(X_1+X_2+X_3+X_4) = -(-2-2-2-6) = 12$ ✓

$A_2 = X_1X_2+Y_1^2+(X_1+X_2)(X_3+X_4)+X_3X_4+Y_3^2$
$= -2(-2) + 0^2 + (-2-2)(-2-6) + (-2)(-6) + 0^2 = 48$ ✓

$A_1 = -[(X_1X_2+Y_1^2)(X_3+X_4) + (X_3X_4+Y_3^2)(X_1+X_2)]$
$= -\{[-2(-2) + 0^2](-2-6) + [(-2)(-6) + 0^2](-2-2)\} = 80$ ✓

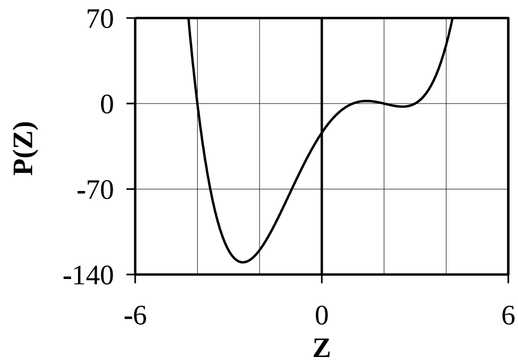
$A_0 = (X_1X_2+Y_1^2)(X_3X_4+Y_3^2) = [-2(-2) + 0^2][-2(-6) + 0^2] = 48$ ✓

Quartic Problem 2: Four Distinct Real Solutions

Problem: Find solutions Z_1, Z_2, Z_3, and Z_4 of

$$\boxed{Z_n^4 - 2Z_n^3 - 13Z_n^2 + 38Z_n - 24 = 0}$$

Given: $A_3 = -2$, $A_2 = -13$, $A_1 = 38$, $A_0 = -24$



Solution with NBS Modified Algorithm:

The NBS resolvent cubic equation is

$$u^3 - A_2u^2 + (A_1A_3 - 4A_0)u + 4A_0A_2 - A_1^2 - A_0A_3^2 = 0 \quad \text{where}$$

$-A_2 = -(-13) = 13$, $\quad A_1A_3 - 4A_0 = 38(-2) - 4(-24) = 20$,

$4A_0A_2 - A_1^2 - A_0A_3^2 = 4(-24)(-13) - 38^2 - (-24)(-2)^2 = -100$.

$\boxed{u^3 + 13u^2 + 20u - 100 = 0}$: Greatest real solution is $u_1 = 2$ from Cubic Problem 2.

$$\Sigma_g = \begin{cases} 1 & \text{if } A_1 - A_3u_1/2 > 0 \\ -1 & \text{otherwise} \end{cases}, \quad A_1 - A_3u_1/2 = 38-(-2)(2)/2 = 40 > 0 \Rightarrow \quad \Sigma_g = 1$$

$$p_1 = A_3/2 - \sqrt{A_3^2/4 + u_1 - A_2} = -2/2 - \sqrt{(-2)^2/4 + 2 - (-13)} = -5$$

$$p_2 = A_3/2 + \sqrt{A_3^2/4 + u_1 - A_2} = -2/2 + \sqrt{(-2)^2/4 + 2 - (-13)} = 3$$

$$q_1 = u_1/2 + \Sigma_g\sqrt{u_1^2/4 - A_0} = 2/2 + (1)\sqrt{2^2/4 - (-24)} = 6$$

$$q_2 = u_1/2 - \Sigma_g\sqrt{u_1^2/4 - A_0} = 2/2 - (1)\sqrt{2^2/4 - (-24)} = -4$$

$$Z_{1,2} = -p_1/2 \pm \sqrt{p_1^2/4 - q_1} = -(-5)/2 \pm \sqrt{(-5)^2/4 - 6} = 2.5 \pm 0.5 \Rightarrow Z_1 = 3,\ Z_2 = 2$$

$$Z_{3,4} = -p_2/2 \pm \sqrt{p_2^2/4 - q_2} = -3/2 \pm \sqrt{3^2/4 - (-4)} = -1.5 \pm 2.5 \Rightarrow Z_3 = 1,\ Z_4 = -4$$

$$\boxed{Z_1 = 3, \quad Z_2 = 2 \quad Z_3 = 1, \quad Z_4 = -4}$$

$(X_1 = 3, \quad X_2 = 2 \quad X_3 = 1, \quad X_4 = -4, \qquad Y_1 = Y_3 = 0)$

Check Solution Against the Input Coefficients: $A_3 = -2$, $A_2 = -13$, $A_1 = 38$, $A_0 = -24$:

$A_3 = -(X_1+X_2+X_3+X_4) = -(3 + 2 + 1 - 4) = -2$✓

$A_2 = X_1X_2+Y_1^2+(X_1+X_2)(X_3+X_4)+X_3X_4+Y_3^2 = 3(2)+0^2+(3+2)(1-4)+1(-4)+0^2 = -13$✓

$A_1 = -[(X_1X_2+Y_1^2)(X_3+X_4) + (X_3X_4+Y_3^2)(X_1+X_2)]$

$= -\{[3(2)+0^2](1-4)+[1(-4)+0^2](3+2) = 38$✓

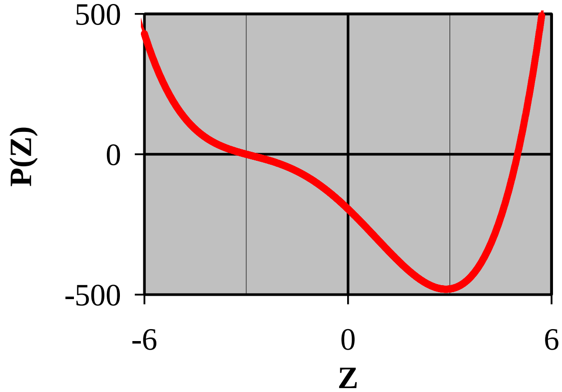
$A_0 = (X_1X_2+Y_1^2)(X_3X_4+Y_3^2) = [3(2)+0^2][(1)(-4)+0^2] = -24$✓

Quartic Problem 3: Two Distinct Real Solutions and a Complex Conjugate Pair

Problem: Find solutions Z_1, Z_2, Z_3, and Z_4 of

$$\boxed{Z_n^4 + 4\,Z_n^3 - 14\,Z_n^2 - 116\,Z_n - 195 = 0}$$

Given: $A_3 = 4,\ A_2 = -14,\ A_1 = -116,$
$A_0 = -195$



Calculation of C, b_2, b_1, and b_0 for both the Euler and the Ferrari modified algorithms:

$C = A_3/4 = 4/4 = 1$

$b_2 = A_2 - 6C^2 = -14 - 6\cdot 1^2 = -20$

$b_1 = A_1 - 2A_2C + 8C^3 = -116 - 2(-14)(1) + 8\cdot 1^3 = -80$

$b_0 = A_0 - A_1C + A_2C^2 - 3C^4 = -195 - (-116)\cdot 1 + (-14)\cdot 1^2 - 3\cdot 1^4 = -96$

Solution with Euler Modified Algorithm:

The Euler resolvent cubic equation is: $r_k^3 + (b_2/2)\, r_k^2 + [(b_2^2 - 4b_0)/16]\, r_k - b_1^2/64 = 0$

with coefficients: $b_2/2 = -20/2 = -10$, $(b_2^2 - 4b_0)/16 = [(-20)^2 - 4(-96)]/16 = 49$,

$-b_1^2/64 = -(-80)^2/64 = -100$.

The resolvent cubic equation becomes $r_k^3 - 10r_k^2 + 49r_k - 100 = 0$.

This is the cubic equation in Cubic Problem 3 above with solutions

$r_1 = 4,\quad r_2 = 3 + 4i,\quad r_3 = 3 - 4i\quad (x_2 = x_3 = 3,\ \ y_2 = -y_3 = 4).$

Also, $b_1 = -80 < 0 \Rightarrow \Sigma = -1$.

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{x_2 + x_3 - 2\Sigma\sqrt{x_2x_3 + y_2^2}} = \sqrt{4} \pm \sqrt{3 + 3 - 2(-1)\sqrt{3\cdot 3 + 4^2}} = 2 \pm 4$$

$$T_{3,4} = -\sqrt{r_1} \pm \sqrt{x_2 + x_3 + 2\Sigma\sqrt{x_2x_3 + y_2^2}} = -\sqrt{4} \pm \sqrt{3 + 3 + 2(-1)\sqrt{3\cdot 3 + 4^2}} = -2 \pm 2i$$

$T_1 = 6,\qquad T_2 = -2,\qquad T_3 = -2 + 2i,\qquad T_4 = -2 - 2i$

$Z_n = T_n - C = T_n - 1,\quad n = 1, 2, 3, 4$

Solution: $\boxed{Z_1 = 5,\quad Z_2 = -3,\quad Z_3 = -3 + 2i,\quad Z_4 = -3 - 2i}$

$(X_1 = 5,\quad X_2 = X_3 = X_4 = -3,\quad Y_1 = Y_2 = 0,\quad Y_3 = -Y_4 = 2)$

<u>Solution with Ferrari Modified Algorithm</u>:

From above we have $C = 1$, $b_2 = -20$, $b_1 = -80$, $b_0 = -96$.

The Ferrari resolvent cubic equation is $m^3 + b_2 m^2 + (b_2^2/4 - b_0)m - b_1^2/8 = 0$ with coefficients: $b_2 = -20$, $b_2^2/4 - b_0 = (-20)^2/4 - (-96) = 196$,

$-b_1^2/8 = -(-80)^2/8 = -800$.

The Ferrari resolvent cubic equation becomes $m^3 - 20m^2 + 196m - 800 = 0$.

Divide it through by 8 and rearrange: $(m/2)^3 - 10(m/2)^2 + 49(m/2) - 100 = 0$.

This is a scaled version of the corresponding Euler resolvent cubic equation on the previous page: $r_k^3 - 10r_k^2 + 49r_k - 100 = 0$ The r_k are replaced by $m/2$. Therefore, the only real solution $r_1 = 4$ from Euler shows that: $m/2 = 4$ or $m = 8$.

Procced with the Ferrari modified algorithm:

$b_1 = -80 < 0 \Rightarrow \Sigma = -1$. $R = \Sigma\sqrt{m^2 + b_2 m + b_2^2/4 - b_0}$

$R = (-1)\sqrt{8^2 + (-20)8 + (-20)^2/4 - (-96)} = -10$

$Z_{1,2} = \sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 - R} = \sqrt{8/2} - 1 \pm \sqrt{-8/2 - (-20)/2 - (-10)}$

$Z_{1,2} = 1 \pm 4$

$Z_{3,4} = -\sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 + R} = -\sqrt{8/2} - 1 \pm \sqrt{-8/2 - (-20)/2 + (-10)}$

$Z_{3,4} = -3 \pm 2i$

Solution: $Z_1 = 5, \quad Z_2 = -3, \quad Z_3 = -3 + 2i, \quad Z_4 = -3 - 2i$

$(X_1 = 5, \quad X_2 = X_3 = X_4 = -3, \quad Y_1 = Y_2 = 0, \quad Y_3 = -Y_4 = 2)$

<u>Check Solution Against the Input Coefficients: $A_3 = 4$, $A_2 = -14$, $A_1 = -116$, $A_0 = -195$</u>:

$A_3 = -(X_1+X_2+X_3+X_4) = -(5 - 3 - 3 - 3) = 4$ ✓

$A_2 = X_1X_2+Y_1^2+(X_1+X_2)(X_3+X_4)+X_3X_4+Y_3^2$

$= 5(-3) + 0^2 + (5-3)(-3-3) + (-3)(-3) + 2^2 = -14$ ✓

$A_1 = -[(X_1X_2+Y_1^2)(X_3+X_4) + (X_3X_4+Y_3^2)(X_1+X_2)]$

$= -\{[5(-3) + 0^2](-3-3) + [(-3)(-3) + 2^2](5 - 3)\} = -116$ ✓

$A_0 = (X_1X_2+Y_1^2)(X_3X_4+Y_3^2) = [5(-3) + 0^2][(-3)(-3) + 2^2] = -195$ ✓

APPENDIX A Example Cubic and Quartic Polynomials and Their Roots

Example cubic and quartic polynomials are plotted to show how the number of real roots is related to the shape of the functional curve. The polynomial coefficients are real numbers, but roots may be complex.

A.1. Cubic Polynomial and its Roots

Every cubic polynomial may be factored as follows:

$$p(z) = z^3 + a_2 z^2 + a_1 z + a_0 = (z - z_1)(z - z_2)(z - z_3) \quad \text{for all } z$$

where z_1, z_2, and z_3 are constants. We can easily verify that z_1, z_2, and z_3 are the roots of $p(z)$. For example, set variable z equal to z_1. Then $(z - z_1)$ becomes zero and so does $p(z)$. In similar fashion, $p(z)$ equals zero if z is set equal to z_2 or z_3. The roots z_1, z_2, and z_3 are functions of the three known real coefficients a_2, a_1, and a_0. Roots z_1, z_2, and z_3 may or may not have values different from each other. Thus, a particular $p(z)$ may have one, two, or three distinct roots. In any case, our problem is to find the three values z_1, z_2, and z_3 given the three real coefficients a_2, a_1, and a_0.

At least one root of a cubic polynomial, root z_1, is real. Additionally, the roots z_2 and z_3 are real, or they are a complex conjugate pair. Thus, suppose z_2 is some complex number $x_2 + iy_2$ where x_2 and y_2 are real, $y_2 \neq 0$, and $i = \sqrt{-1}$. Then z_3 is the complex number $x_2 - iy_2$. Whether z_2 and z_3 are real or not, finding the roots of the cubic $p(z)$ requires that we find three real numbers. If z_1, z_2, and z_3 are all real, then the three real numbers are z_1, z_2, and z_3. If only z_1 is real, then the three real numbers are z_1, x_2, and y_2. This paper uses the convention that if z_1, z_2, and z_3 are all real, then z_1 is the root of greatest value.

A.1.1. Examples of the Cubic Polynomial and its Roots

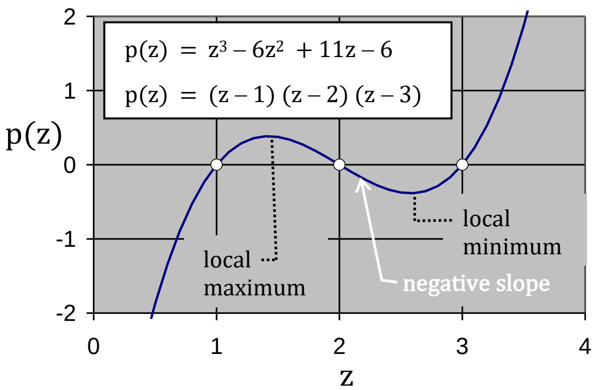
Figure A-1 plots examples of cubic polynomial $p(z)$. Even though we consider roots that are complex as well as real, the plots show only real z and the corresponding real $p(z)$. Figure A-1(a) plots a cubic having three distinct real roots; $z_1 = 1$, $z_2 = 2$, and $z_3 = 3$. In each of Figure A-1(b) and (c), the three roots are all real, but they are not distinct. That is, the roots do not have three different values. The example in Figure A-1(b) has only two distinct roots: 1 and 3. The factor $(z - 1)$ appears twice when the polynomial is factored:

$$z^3 - 5z^2 + 7z - 3 \;=\; (z-1)(z-1)(z-3) \;=\; (z-1)^2(z-3)$$

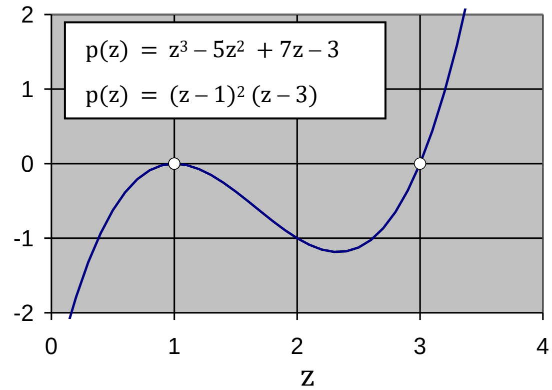
We say that the root 1 has a *multiplicity* of two. The root 3 is a *simple root* (multiplicity of one).

The example in Figure A-1(c) has only the one distinct root 2, but it has a multiplicity of three:

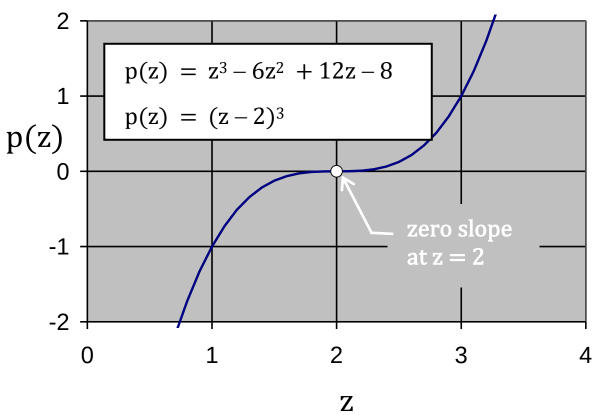
$$z^3 - 6z^2 + 12z - 8 \;=\; (z-2)(z-2)(z-2) \;=\; (z-2)^3$$



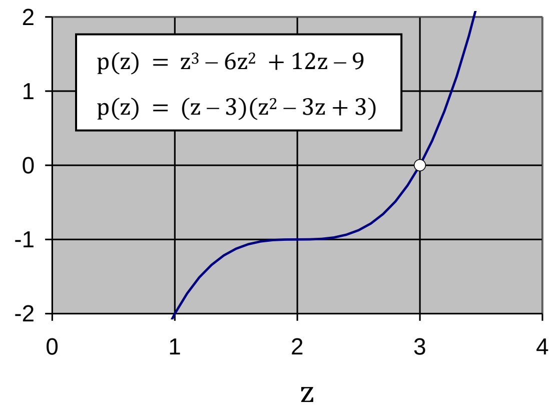
(a) Three Distinct Real Roots: 1, 2, and 3



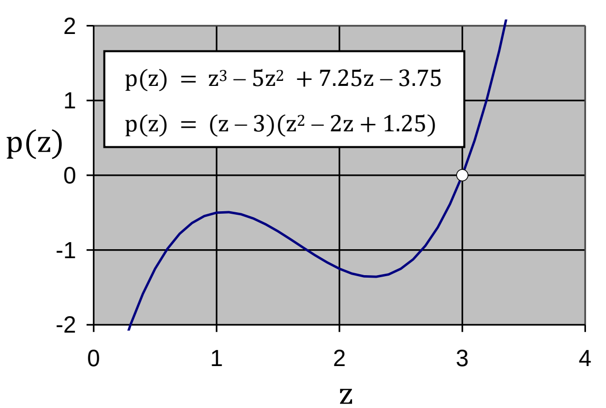
(b) Two Distinct Real Roots: 1 and 3
Root 1 has a multiplicity of 2



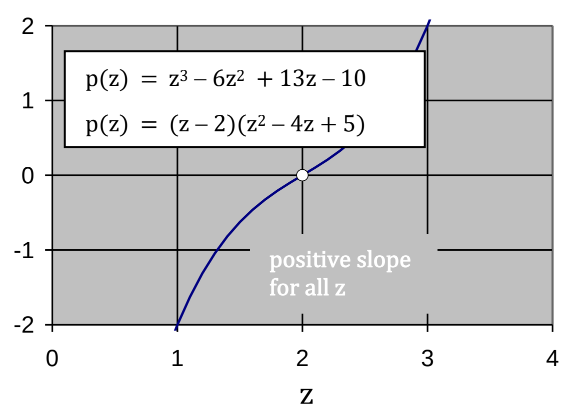
(c) Real Root 2 with Multiplicity of 3



(d) Real Root 3 and Complex Conjugate Roots $3/2 \pm i\sqrt{3}/2$



(e) Real Root 3 and Complex Conjugate Roots $1 \pm i/2$



(f) Real Root 2 and Complex Conjugate Roots $2 \pm i$

Figure A-1 Plots of Example Cubic Polynomial Functions

The cubic $z^3 - 6z^2 + 12z - 8$ from Figure A-1(c) is plotted again in Figure A-1(d) with the additive constant a_0 changed from –8 to –9. This new polynomial $p(z) = z^3 - 6z^2 + 12z - 9$ now has one real root with a value of $z_1 = 3$. If we divide $z - 3$ into $z^3 - 6z^2 + 12z - 9$, the quotient is the quadratic $z^2 - 3z + 3$. That is:

$$z^3 - 6z^2 + 12z - 9 \quad = \quad (z - 3)(z^2 - 3z + 3).$$

The roots z_2 and z_3 of cubic $z^3 - 6z^2 + 12z - 9$ are the roots of the quadratic factor $z^2 - 3z + 3$. They can be found using the quadratic formula:

$$\left.\begin{matrix} z_2 \\ z_3 \end{matrix}\right\} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{3^2 - 4(1)(3)}}{2} = \frac{3 \pm \sqrt{-3}}{2} = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}.$$

Thus, the three roots of $p(z) = z^3 - 6z^2 + 12z - 9$ are $z_1 = 3$ and the complex conjugate pair $z_2 = 3/2 + i\sqrt{3}/2$ and $z_3 = 3/2 - i\sqrt{3}/2$.

Figure A-1(e) and (f) show two more cubic polynomials that have one real root and a pair of complex conjugate roots.

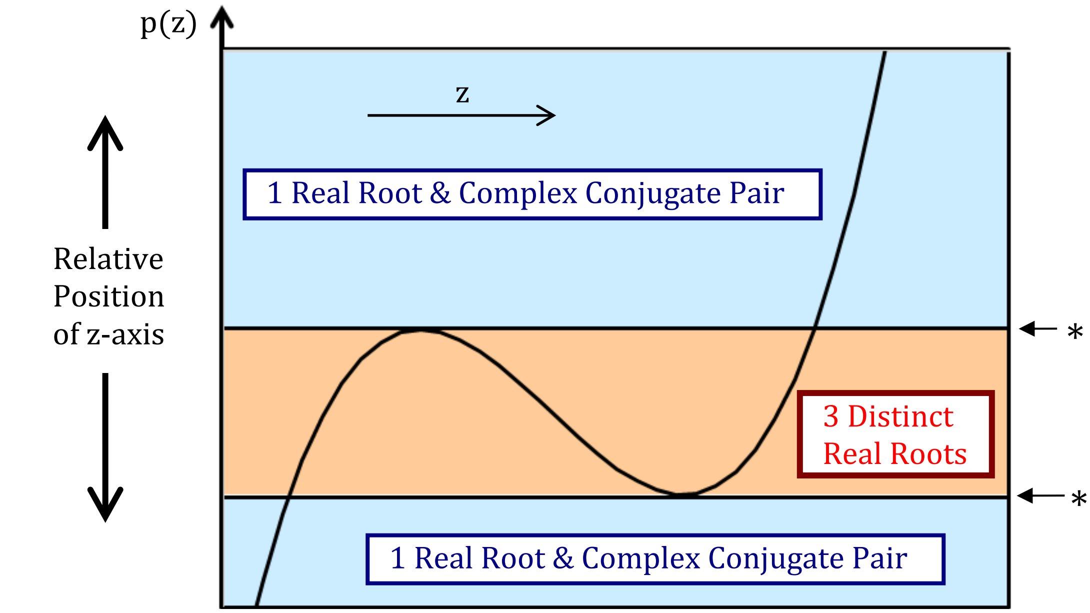
A.1.2. The Three Basic Shapes of Cubic Polynomial Functions

All cubic polynomials share some common features for real z:

- As z approaches $-\infty$, p(z) approaches $-\infty$.
- As z approaches $+\infty$, p(z) approaches $+\infty$.
- A tangent to the function curve has a positive slope for large negative z values and for large positive z values. That is, for large z values of either sign, a slight increase in the value of z produces an increase in the value of p(z).

All cubic polynomial function curves have one of three basic shapes as shown in Figure A-1. The first basic shape is illustrated in Figure A-1(a), (b), and (e). The cubic has a local maximum and a local minimum. For z values between the local maximum and minimum, a tangent to the curve has a negative slope. In the three examples shown here, the negative slope occurs near $z = 2$. A slight *increase* in the value of z produces a *decrease* in the value of p(z). Only cubic functions with such a region of negative slope can have multiple, discrete real roots.

For this first shape, Figure A-2 illustrates how the position of the z-axis, relative to the local maximum and minimum, determines the type of roots the polynomial has. The polynomial has one real root and a complex conjugate pair of roots if the z-axis falls either below the local minimum or above the local maximum. That is, either the local minimum value of p(z) is greater than zero or the local maximum value of p(z) is less than zero, as in Figure A-1(e).



* 2 Distinct Real Roots: Root at Relative Maximum or Minimum Has Multiplicity 2.

Figure A-2   Cubic Polynomial with Local Maximum and Minimum

The polynomial has three distinct real roots if the z-axis falls between the local minimum and local maximum p(z) values, as in Figure A-1(a). Finally, the polynomial has two distinct real roots if the z-axis falls exactly on the local maximum (Figure A-1 (b)) or local minimum. The root corresponding to the local maximum or minimum has a multiplicity of two.

Figure A-1(c) and (d) illustrate the second basic shape of a cubic polynomial: the curve has a positive slope for all but one value of z. At that one value, the slope is zero: the tangent to the curve is perfectly horizontal. In both of our examples, the zero slope occurs at $z = 2$. If and only if the zero slope occurs at a real root, as in Figure A-1(c), the root will be the only root and have a multiplicity of three. Otherwise there will be a real root and a pair of complex conjugate roots as in Figure A-1(d).

Figure A-1(f) illustrates the third basic shape of a cubic polynomial: the curve has a positive slope for all z values. A cubic function of this shape always has one real root and a pair of complex conjugate roots.

The following table shows how the coefficients a_2 and a_1 determine the basic shape of a cubic.

DETERMINING THE BASIC FUNCTIONAL SHAPE OF CUBIC $p(z) = z^3 + a_2z^2 + a_1z + a_0$			
Condition	Basic Functional Shape	Figure A-1 Examples	Number of Discrete Real Roots
$a_2^2 - 3a_1 > 0$	Tangent slope is negative between a local maximum and local minimum.	(a), (b), (e)	3, 2, or 1
$a_2^2 - 3a_1 = 0$	Tangent slope is zero at one z value. Slope is positive otherwise.	(c), (d)	1
$a_2^2 - 3a_1 < 0$	Tangent slope is greater than zero for all z.	(f)	1

If $a_2^2 - 3a_1 > 0$, then p(z) has a local maximum at $z = \frac{1}{3}\left(-a_2 - \sqrt{a_2^2 - 3a_1}\right)$ and a local minimum at $z = \frac{1}{3}\left(-a_2 + \sqrt{a_2^2 - 3a_1}\right)$.

If $a_2^2 - 3a_1 = 0$, then p(z) has zero slope at $z = -a_2/3$.

Regardless of the basic functional shape, p(z) has a point of inflection at $z = -a_2/3$. For z values less than $-a_2/3$, the slope of the functional curve decreases with increasing z, and p(z) curves down (negative curvature). For z values greater than $-a_2/3$, the slope of the functional curve increases with increasing z, and p(z) curves up (positive curvature). Thus the point of inflection is the z value at which p(z) transitions from negative curvature to positive curvature.

A.2. Quartic Polynomial and its Roots

Any quartic polynomial P(Z) with real coefficients can be expressed as the product of two quadratic functions each having real coefficients:

$$P(Z) = Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z^2 + B_1Z + C_1)(Z^2 + B_2Z + C_2) \quad \text{for all } Z$$

Some analytic algorithms for finding the roots of a quartic use this property of quartic functions. Given a quartic function, the algorithms find a corresponding pair of quadratic factors. That is the hard part. The roots of the two quadratics are easily found via the quadratic formula. The roots of the two quadratic factors are the roots of P(Z). If Z_1 and Z_2 are the roots of the first quadratic factor and if Z_3 and Z_4 are the roots of the second quadratic factor, then P(Z) has the four roots Z_1, Z_2, Z_3, and Z_4.

P(Z) can be factored as:

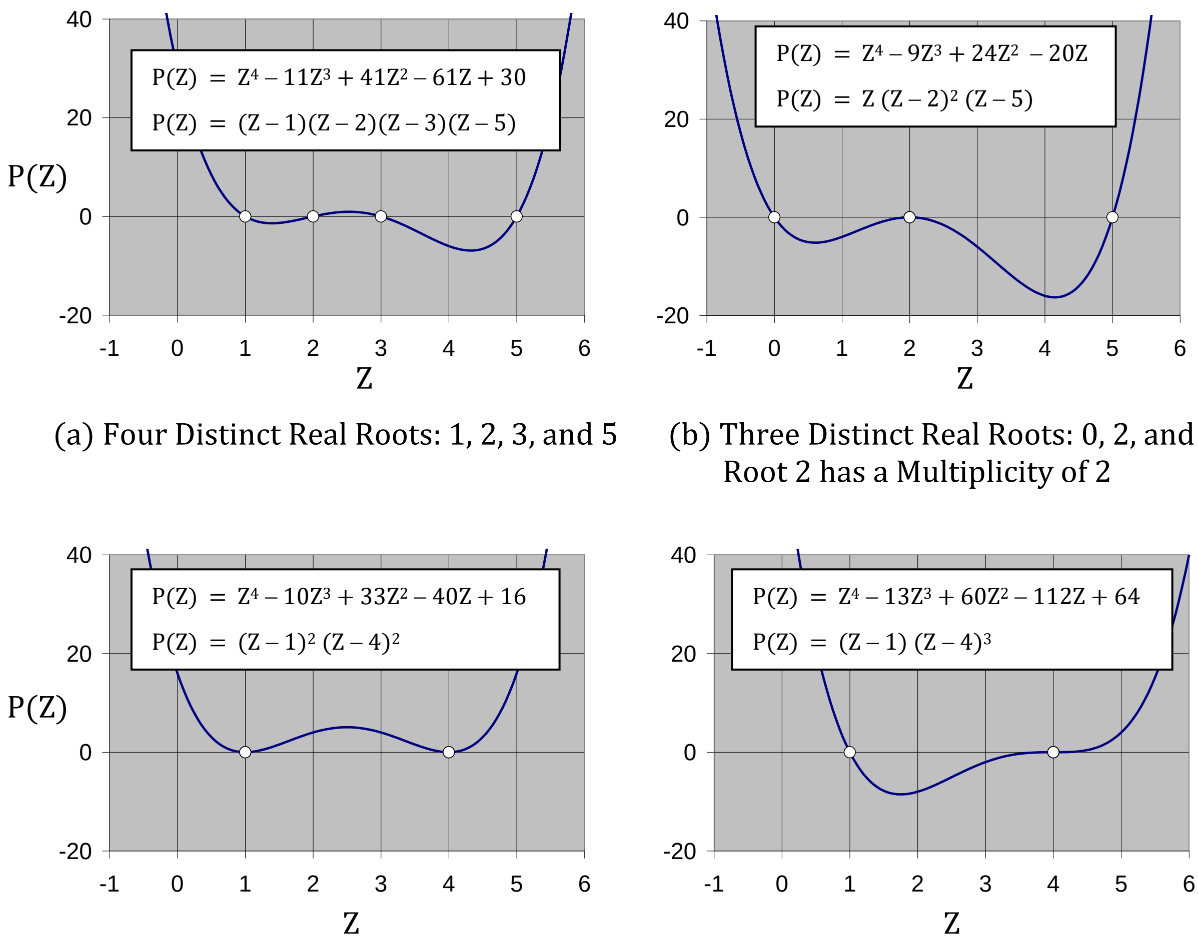
$$P(Z) = Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

We index the roots so that Z_1 and Z_2 are either real or they are a complex conjugate pair. Roots Z_3 and Z_4 are either real or they are a complex conjugate pair. A pair of real roots may or may not have the same value.

As shown below, the pair of quadratic factors for a given P(Z) is not always unique; sometimes a particular P(Z) can be factored into a pair of quadratic factors in two or three different ways.

A.2.1. Examples of the Quartic Polynomial and its Roots

Figure A-3 below plots some example quartic polynomials P(Z) versus real Z.



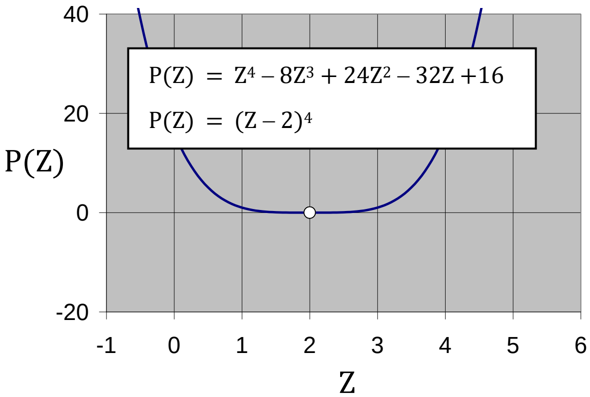
(a) Four Distinct Real Roots: 1, 2, 3, and 5

(b) Three Distinct Real Roots: 0, 2, and 5
Root 2 has a Multiplicity of 2

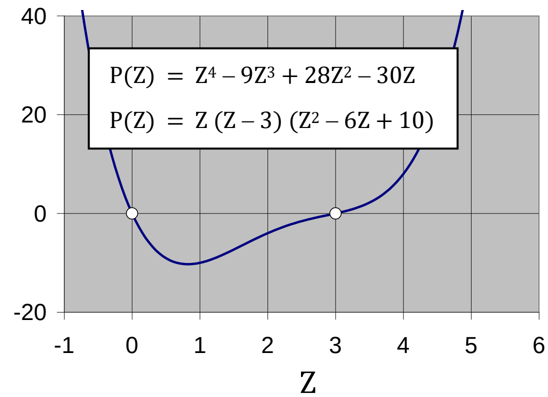
(c) Two Distinct Real Roots: 1 and 4,
Each with a Multiplicity of 2

(d) Two Distinct Real Roots: 1 and 4,
Root 4 has a Multiplicity of 3

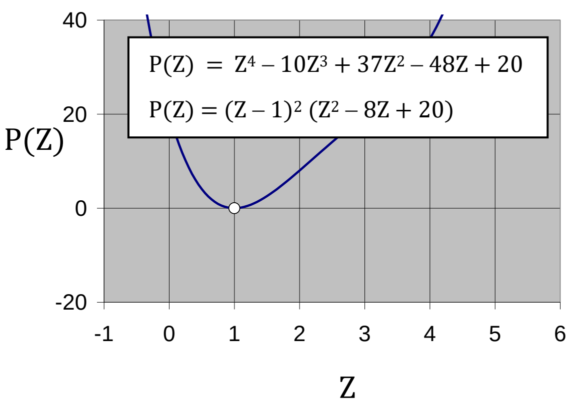
Figure A-3 Plots of Example Quartic Polynomial Functions
(Page 1 of 2)



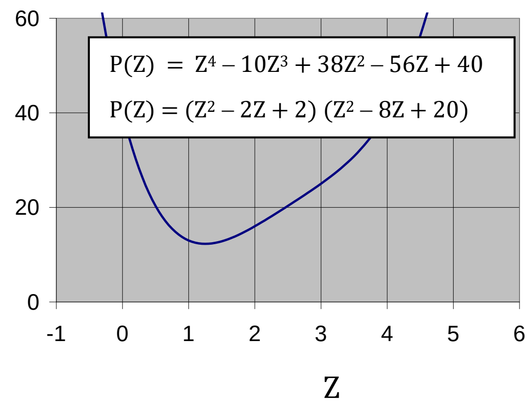
(e) Real Root 2 with Multiplicity of 4



(f) Real Roots 0, 3, and Complex Conjugate Roots $3 \pm i$



(g) Real Root 1 with Multiplicity 2 and Complex Conjugate Roots: $4 \pm 2i$



(h) Complex Conjugate Roots: $1 \pm i$ and $4 \pm 2i$

Figure A-3 Plots of Example Quartic Polynomial Functions
(Page 2 of 2)

Figure A-3(a) shows an example quartic having four distinct real roots: 1, 2, 3, and 5. This quartic polynomial $P(Z) = Z^4 - 11Z^3 + 41Z^2 - 61Z + 30$ can be expressed as the product of a pair of quadratics in three different ways corresponding to the three possible pairings of the four distinct roots:

$$P(Z) = [(Z-1)(Z-2)]\,[(Z-3)(Z-5)] = (Z^2 - 3Z + 2)\,(Z^2 - 8Z + 15)$$

$$P(Z) = [(Z-1)(Z-3)]\,[(Z-2)(Z-5)] = (Z^2 - 4Z + 3)\,(Z^2 - 7Z + 10)$$

$$P(Z) = [(Z-1)(Z-5)]\,[(Z-2)(Z-3)] = (Z^2 - 6Z + 5)\,(Z^2 - 5Z + 6)$$

Only a P(Z) with four distinct real roots can equal any of three different pairs of quadratics in this way. Expressing the original quartic P(Z) as any one of these three quadratic products will produce the same four quartic roots.

In Figure A-3(b), quartic $P(Z) = Z^4 - 9Z^3 + 24Z^2 - 20Z$ has three distinct real roots: 0, 2, and 5. Root 2 has a multiplicity of two. This P(Z) can be expressed as the product of a pair of quadratics in two different ways corresponding to whether the root 2 is paired with itself or not:

$$P(Z) = [(Z-2)(Z-2)]\,[Z(Z-5)] = (Z^2 - 4Z + 4)\,(Z^2 - 5Z)$$

$$P(Z) = [Z(Z-2)]\,[(Z-2)(Z-5)] = (Z^2 - 2Z)\,(Z^2 - 7Z + 10).$$

Figure A-3(c) plots $P(Z) = Z^4 - 10Z^3 + 33Z^2 - 40Z + 16$, which has two distinct real roots, 1 and 4, each with a multiplicity of two. Again, there are two possible quadratic factors corresponding to whether each distinct root is paired with itself or not:

$$P(Z) = [(Z-1)(Z-1)]\,[(Z-4)(Z-4)] = (Z^2 - 2Z + 1)\,(Z^2 - 8Z + 16)$$

$$P(Z) = [(Z-1)(Z-4)]\,[(Z-1)(Z-4)] = (Z^2 - 5Z + 4)\,(Z^2 - 5Z + 4).$$

Figure A-3(d) plots $P(Z) = Z^4 - 13Z^3 + 60Z^2 - 112Z + 64$. It also has the real roots 1 and 4, but in this case, the root 1 is a simple root (multiplicity of 1) and the root 4 has a multiplicity of three. There is only one quadratic factorization:

$$P(Z) = [(Z-1)(Z-4)]\,[(Z-4)(Z-4)] = (Z^2 - 5Z + 4)\,(Z^2 - 8Z + 16).$$

Figure A-3(e) plots $P(Z) = Z^4 - 8Z^3 + 24Z^2 - 32Z + 16$, which has only one real root, root 2, with a multiplicity of four. The only quadratic factorization is:

$$P(Z) = [(Z-2)(Z-2)]\,[(Z-2)(Z-2)] = (Z^2 - 4Z + 4)^2.$$

Figure A-3(f) plots $P(Z) = Z^4 - 9Z^3 + 28Z^2 - 30Z$, which has a real pair of roots (0 and 3) and a complex conjugate pair of roots ($3 + i$ and $3 - i$). Any root with a nonzero imaginary component must be paired with its complex conjugate to form a quadratic with real coefficients. The only quadratic factorization for Figure A-3(f) is:

$$P(Z) = [Z(Z-3)]\,\{[Z-(3+i)][Z-(3-i)]\} = (Z^2 - 3Z)\,(Z^2 - 6Z + 10).$$

Figure A-3(g) plots $P(Z) = Z^4 - 10Z^3 + 37Z^2 - 48Z + 20$, which has real root 1 with multiplicity of two and a complex conjugate pair of roots ($4 + 2i$ and $4 - 2i$). The only quadratic factorization is:

$$P(Z) = [(Z-1)^2]\,\{[Z-(4+2i)][Z-(4-2i)]\} = (Z^2 - 2Z + 1)\,(Z^2 - 8Z + 20).$$

Figure A-3(h) plots the quartic $P(Z) = Z^4 - 10Z^3 + 38Z^2 - 56Z + 40$, which has two different pair of complex conjugate roots: $1 \pm i$ and $4 \pm 2i$. The quadratic factorization is:

$$\begin{aligned} P(Z) &= \{[Z-(1+i)][Z-(1-i)]\}\{[Z-(4+2i)][Z-(4-2i)]\} \\ &= (Z^2 - 2Z + 2)(Z^2 - 8Z + 20). \end{aligned}$$

A.2.2. Shapes of Quartic Polynomial Functions

The shape of a particular quartic polynomial P(Z) versus real Z, like that of a cubic, can be related to the polynomial's possible combinations of real and complex roots.

All quartic polynomials P(Z) share some common features for real Z:

- P(Z) approaches $+\infty$ as z approaches either $-\infty$ or $+\infty$;
- P(Z) has a finite minimum value;
- P(Z) has either one or two local minima; and
- The derivative $P'(Z)$ equals zero at 1, 2, or 3 discrete values of Z. The derivative $P'(Z)$ is the slope of the functional curve.

Quartics with Two Local Minima

Figures A-3(a), (b), and (c) are examples of quartics with two local minima. For such P(Z), there are three discrete Z values at which the derivative $P'(Z)$ equals zero. Two of the Z values are those corresponding to the two local minima. The third lies between the first two and corresponds to a local maximum of P(Z).

If the P(Z) values of the two local minima are different from each other, then the P(Z) may have 0, 1, 2, 3, or 4 distinct real roots. Figure A-4 illustrates how the position of the Z-axis, relative to the local minima and local maximum, determines whether the roots are complex conjugate pairs, discrete real roots, or real roots with multiplicity. Consider the green region in the figure as an example. In this case the Z-axis lies above the greater local minimum and below the local maximum of P(Z). In other words, the greater local minimum of P(Z) is less than zero and the local maximum of P(Z) is greater than zero. Such a P(Z) has four discrete real roots, just as in Figure A-3(a).

A special case occurs when the two local minima have the same P(Z) value. The yellow region in Figure A-4 vanishes. If these two local minima have a P(Z) value of zero, then the corresponding Z values are discrete real roots, each having a multiplicity of two, just as in Figure A-3(c).

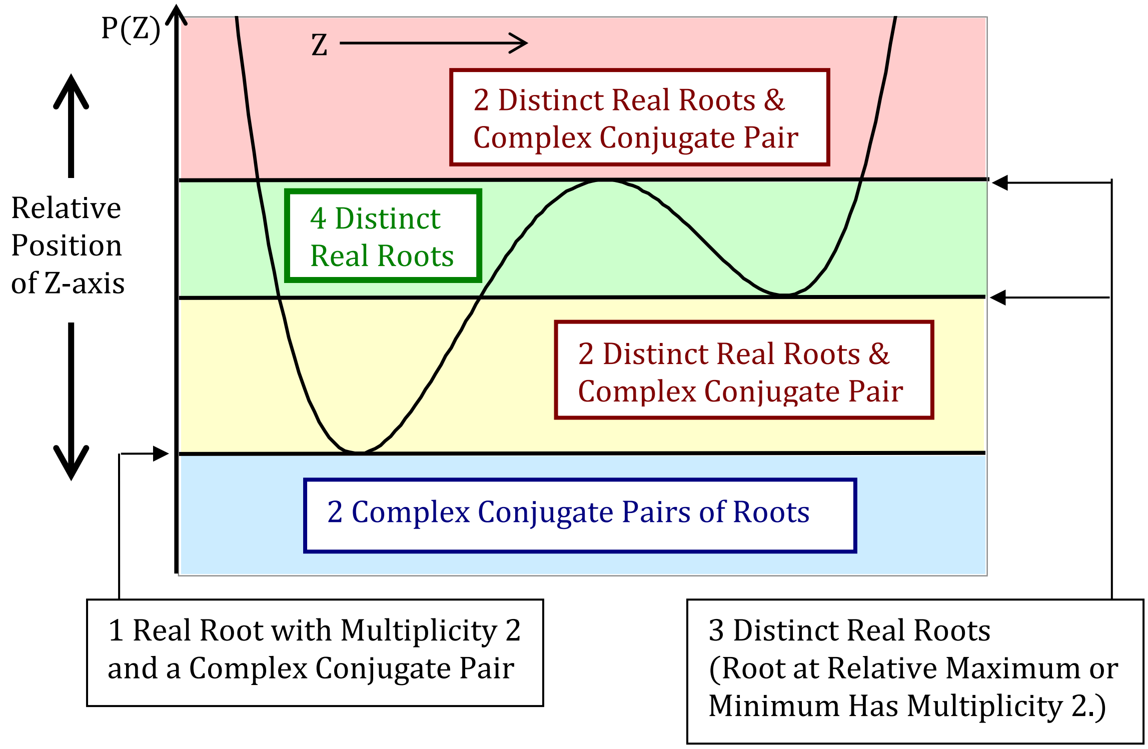


Figure A-4 Quartic Polynomial with Two Local Minima

Quartics with One Local Minimum

Figures A-3(d) through (h) are all examples of quartics with only one local minimum. The local minimum is also the global minimum. For such P(Z), there are either one or two Z values at which the derivative P′(Z) equals zero. In Figures A-3(e) through (h), P(Z) has only one Z value for which $P'(Z) = 0$. That Z value corresponds to the P(Z) minimum.

Figure A-3(d) is an example in which P′(Z) equals zero at two Z values: $Z = 1.75$ and $Z = 4$. The first of these, $Z = 1.75$, is the quartic's minimum value. The second, $Z = 4$, is a point of inflection. A point of inflection is a point at which the curvature of the function changes signs. Our example function is concave upwards (positive curvature) for Z values less than 2.5 and for Z values greater than 4. P(Z) is concave downwards (negative curvature) for Z between 2.5 and 4. Therefore $Z = 2.5$ and $Z = 4$ are points of inflection. P(Z) has its particular shape (one local minimum but $P'(Z) = 0$ in two places) because $Z = 4$ is both a point inflection and a point of zero slope. Since $Z = 4$ is also a real root, this example P(Z) represents a special class of quartic with the following property. If and only if a quartic P(Z) has a real root Z_1 such that derivative $P'(Z_1) = 0$ and Z_1 is a point of inflection, then P(Z) has two distinct real roots, Z_1 and Z_2, where root Z_1 has multiplicity of three. Thus P(Z) has the form $P(Z) = (Z - Z_1)^3 (Z - Z_2)$. In our example, $Z_1 = 4$ and $Z_2 = 1$.

The quartics in Figures A-3(e) and (g) each have a real root at the function's only local minimum, and that real root is the only Z value at which $P'(Z) = 0$. The real root either has a multiplicity of four as in Figure A-3(e) or a multiplicity of two as in Figures A-3(g). If the multiplicity is two, then P(Z) also has a complex conjugate pair of roots.

Finally, we have the cases in which there is no real root corresponding to a zero slope. If the minimum P(Z) is less than zero, then there are two distinct real roots and a complex conjugate pair of roots as in Figures A-3(f). If the minimum P(Z) is greater than zero, then there are two complex conjugate pairs of roots as in Figures A-3(h).

The following table shows how the coefficients A_3, A_2, and A_1 determine the basic shape of a quartic.

DETERMINING THE BASIC FUNCTIONAL SHAPE OF A QUARTIC: $P(Z) = Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0$			
$q = (8A_2 - 3A_3^2)/48$	$r = (4A_2A_3 - 8A_1 - A_3^3)/64$		
Condition	Basic Functional Shape	Figures A-3 Examples	Number of Discrete Real Roots
$r^2 + q^3 < 0$	two local minima	(a), (b), (c)	4, 3, 2, 1, or 0
$r \neq 0$, $r^2 + q^3 = 0$	one local minimum and point of inflection with zero slope	(d)	2, 1, or 0
$r = q = 0$	one local minimum, U-shape functional curve	(e)	2, 1, or 0
$r^2 + q^3 > 0$	one local minimum	(f), (g), (h)	2, 1, or 0

APPENDIX B Review of Requisite Mathematics

This review covers the mathematics needed to use and derive the analytic algorithms for solving cubic and quartic equations. The review includes properties of polynomials, trigonometry concepts, and properties of complex numbers.

B.1. Properties of Polynomials

The following paragraphs provide a review of some properties that apply to single-variable polynomials with real coefficients. A polynomial $p_n(z)$ of *degree* n in the single variable z has the form

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where the coefficients a_i are real, $a_n \neq 0$, and integer $n \geq 1$. Variable z may be real but is generally complex.

B.1.1. Equality

Two polynomials are equal to each other if and only if they have the same degree and the corresponding coefficients of the two polynomials are equal to each other. That is, if $p_n(z)$ and $q_m(z)$ are polynomials where

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad \text{and}$$

$$q_m(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0,$$

then $p_n(z) = q_m(z)$ if and only if

$$n = m, \quad a_n = b_n, \quad a_{n-1} = b_{n-1}, \quad \cdots, \quad a_1 = b_1, \quad \text{and} \quad a_0 = b_0.$$

B.1.2. Factor Theorem

If and only if z_r is a root of $p_n(z)$, then $z - z_r$ is a factor of $p_n(z)$. That is,

$$p_n(z_r) = 0 \quad \Leftrightarrow \quad p_n(z) = (z - z_r)\, q_m(z)$$

where $q_m(z)$ is a polynomial of degree m less than n.

B.1.3. Linear Factors

Every polynomial $p_n(z)$ has n linear factors (not necessarily distinct) such that

$$p_n(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

where $z_1, z_2, \ldots, z_n$ are numbers, generally complex. These n numbers are the n roots (not necessarily distinct) of $p_n(z)$.

B.1.4. Conjugate Pairs Theorem

If polynomial $p_n(z)$ has a complex root $z_r = x_r + iy_r$ where x_r and y_r are each real with $y_r \neq 0$, then the complex conjugate $z_r^* = x_r - iy_r$ is also a root of $p_n(z)$.

B.1.5. Corollary for Polynomials of Odd Degree

If the degree n of polynomial $p_n(z)$ is odd, then $p_n(z)$ has at least one real root.

Note: Properties B.1.1, B.1.2, and B.1.3 apply to polynomials with complex coefficients as well as to those with real coefficients.

B.2. Trigonometry Concepts

The following paragraphs review radian measure, trigonometric functions, selected trigonometric identities, and inverse trigonometric functions. The radian measure of an angle, as well as the trigonometric functions, may be defined in reference to a circle centered at the origin of an x,y Cartesian coordinate system, Figure B-1.

B.2.1. Radian Measure of an Angle

Let r be the radius of the circle. Letter A denotes the point (r,0) at which the circle intersects the positive x-axis, B denotes the origin, and C denotes a general point (x,y) on the circle.

The radian measure θ of angle ∠ABC has the magnitude

$$|\theta| = \frac{S}{r}$$

where S is the arc length of the circular arc $\overset{\frown}{AC}$. Positive θ values are measured from the positive x-axis in the counter-clockwise sense. Thus θ in Figure B-1 is positive.

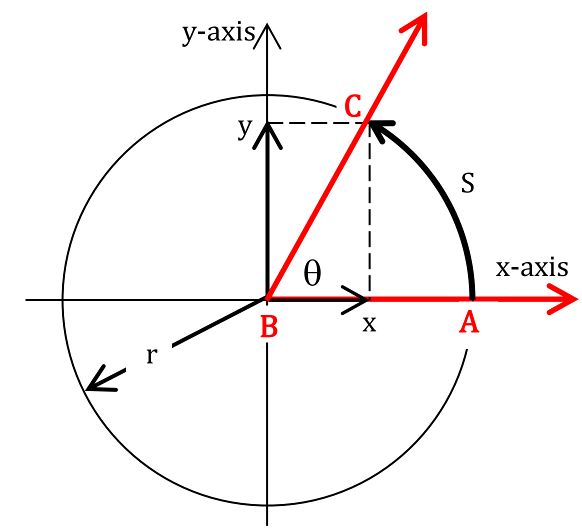
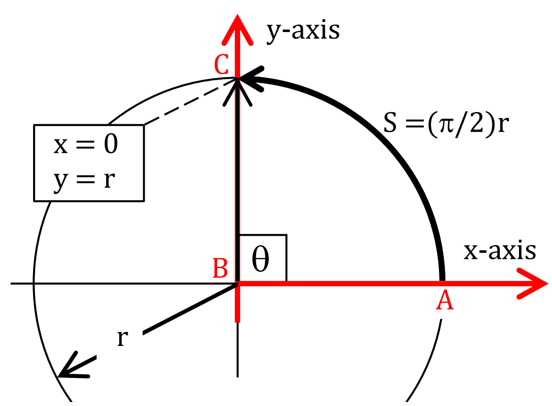


Figure B-1 Geometry for Defining Radian Measure and Trigonometric Functions

Consider the following case of a right angle: x = 0, y = r. Then point C is the intersection of the circle with the positive y-axis. The angle measured from the positive x-axis is a right angle, and the arc length S is ¼ of the circle's circumference: $S = 2\pi r/4 = (\pi/2)r$. Therefore, $\theta = +S/r = \pi/2$. The radian measure of a 90-degree angle (right angle) is $\pi/2$.



Special Case of a Right Angle

Similarly, the radian measure of a 180-degree angle (straight angle) is π.

180 degrees = π radians = ½ revolution

The radian measure of an angle is unitless because it is defined as the ratio of two lengths, S and r. In practice, however, the measure θ of an angle is often said to have units of radians in order to distinguish it from an angle measure in degrees.

The principal range of θ is $(-\pi,\pi]$.* That is, θ is in the principal range if $-\pi < \theta \le \pi$. There is a unique value of θ in this range for every angle and the corresponding point (x,y) of the circle.

A radian measure of $\theta \pm 2\pi n$ for any integer n also applies to the angle. This is so because the circular arc can be extended any number of complete revolutions in either the counter-clockwise or clockwise direction and the arc still terminates at the same point C = (x,y). The circumference of a circle is $2\pi r$, so each revolution has an arc length of $2\pi r$ and a radian measure of 2π. Unless noted otherwise, the radian measure under discussion is assumed to fall within the principal range $(-\pi,\pi]$.

In trigonometry, we often refer to θ as an angle rather than using the more formal terminology: θ is the measure of some angle ∠ABC. This relaxed terminology is permissible provided that the context presents no ambiguity.

B.2.2. Trigonometric Functions

The adjacent chart defines the trigonometric functions of θ with respect to Figure B-1. These functions are all periodic.

Name	Definition	Name	Definition
sine	$\sin\theta = y/r$	cosecant	$\csc\theta = r/y$
cosine	$\cos\theta = x/r$	secant	$\sec\theta = r/x$
tangent	$\tan\theta = y/x$	cotangent	$\cot\theta = x/y$

The sine and cosine functions are defined for all real values of θ and have periods of 2π. They are plotted versus θ over two periods in Figure B-2. The sine of θ is an odd function ranging from a minimum of –1 at $\theta = -\pi/2$ to a maximum of 1 at $\theta = \pi/2$. The cosine of θ is even with minimum –1 at $\theta = \pm\pi$ and maximum 1 at $\theta = 0$. Note that a function f(x) is even if $f(-x) = f(x)$; it is odd if $f(-x) = -f(x)$.

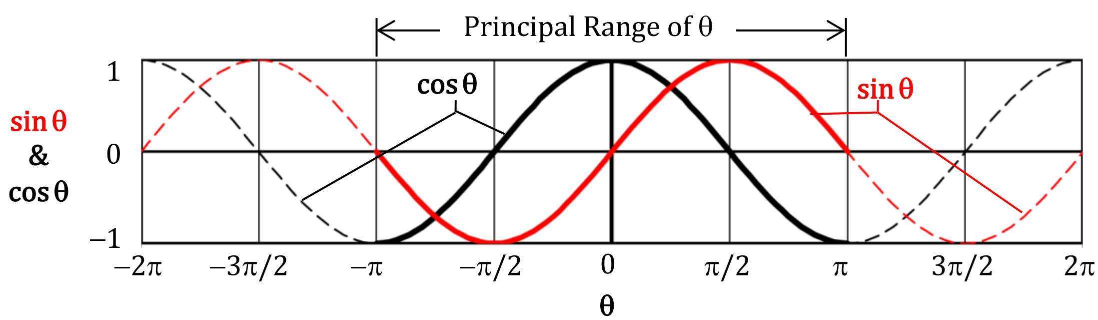


Figure B-2 Plot of Sine and Cosine Functions

* Other authors sometimes use a principal range of $[0,2\pi)$.

Values of sin θ, cos θ, and tan θ for Selected θ Values

θ - deg	0°	30°	45°	60°	90°	120°	135°	150°	180°
θ - rad	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
sin θ	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
sin(–θ)	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	–1	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0
cos θ	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	–1
cos(–θ)	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	–1
tan θ	0	$\sqrt{3}/3$	1	$\sqrt{3}$	---	$-\sqrt{3}$	–1	$-\sqrt{3}/3$	0
tan(–θ)	0	$-\sqrt{3}/3$	–1	$-\sqrt{3}$	---	$\sqrt{3}$	1	$\sqrt{3}/3$	0

Figure B-3 below plots csc θ (reciprocal of sin θ) and sec θ (reciprocal of cos θ). Like sin θ, csc θ is an odd function with period 2π. The cosecant, however, is undefined for θ equal to integer multiples of π (θ = nπ), for which sin θ is zero. Like cos θ, sec θ is an even function with period 2π. The secant, however, is undefined for θ equal to odd-integer multiples of π/2 ($\theta = \frac{2n+1}{2}\pi$), for which cos θ is zero.

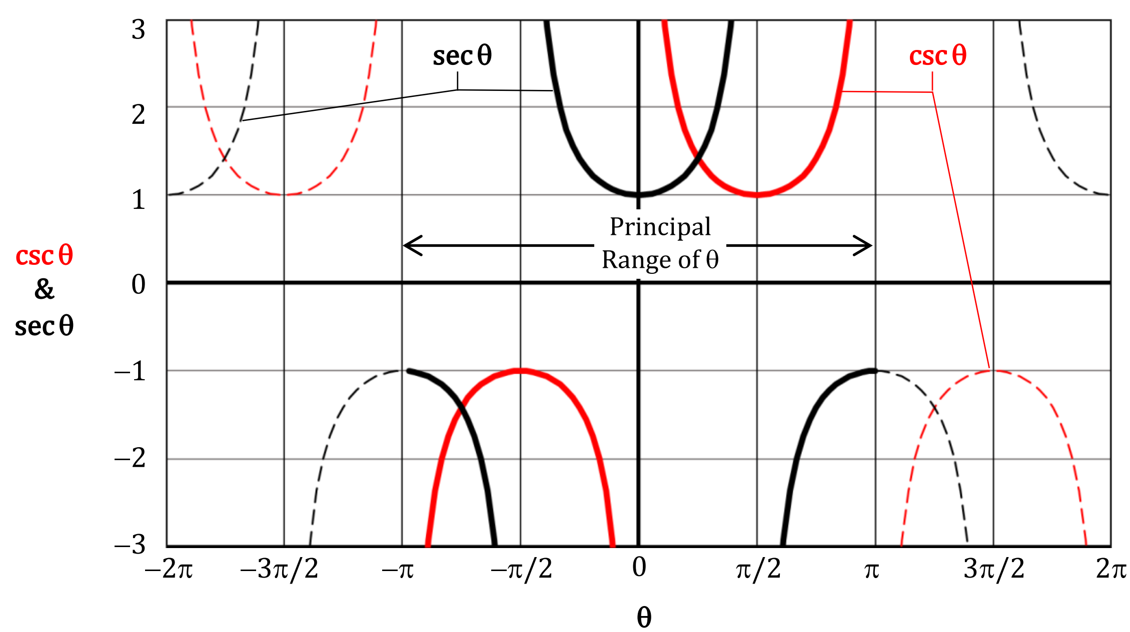


Figure B-3 Plot of Cosecant and Secant Functions

Figure B-4 below plots tan θ and cot θ. Each of these functions is odd and has a period of π. The tangent may be expressed as tan θ = sin θ/cos θ. Therefore, tan θ is zero for θ equal to integer multiples of π, for which sin θ = 0 and cos θ = ±1. Also, tan θ is undefined for θ equal to odd-integer multiples of π/2, for which cos θ = 0 and sin θ = ±1. The cotangent is the reciprocal of the tangent: cot θ = 1/tan θ = cos θ/sin θ.

Therefore, $\cot\theta$ is zero for θ equal to odd-integer multiples of $\pi/2$, and $\cot\theta$ is undefined for θ equal to integer multiples of π.

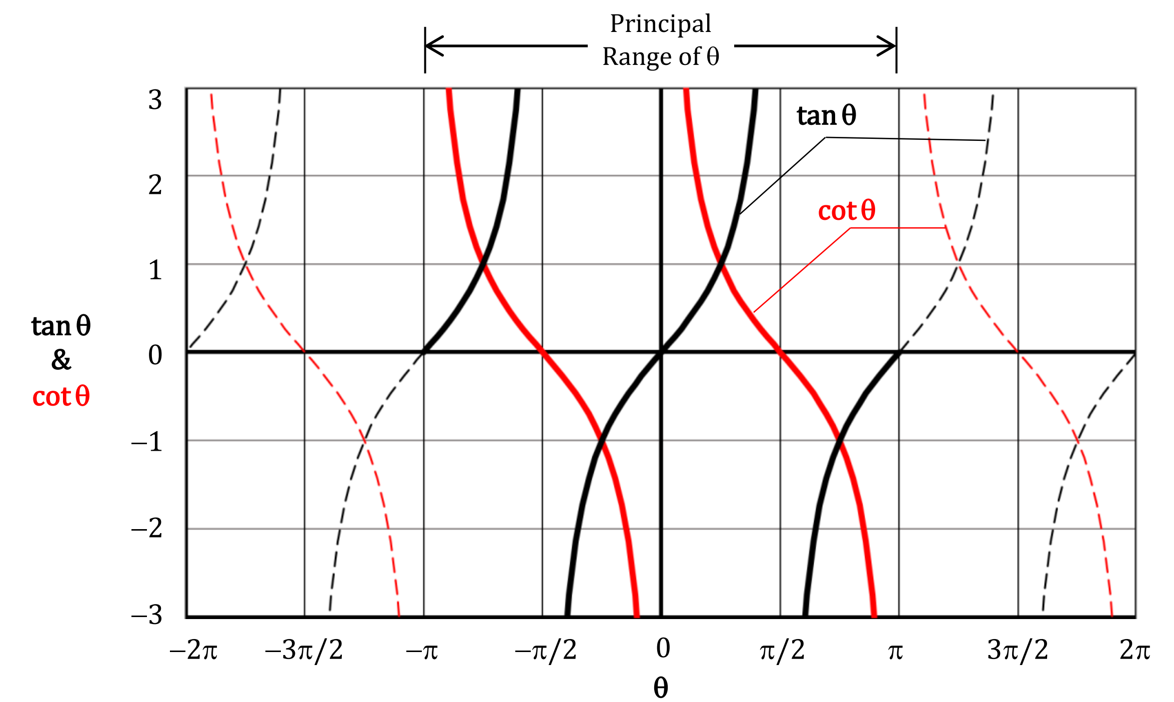


Figure B-4 Plot of Tangent and Cotangent Functions

B.2.3. Selected Trigonometric Identities

In the following tables, n is an integer.

Relations of the Trigonometric Functions

$\csc\theta = 1/\sin\theta$ $\quad$ $\sec\theta = 1/\cos\theta$ $\quad$ $\cot\theta = 1/\tan\theta$ $\quad$ $\tan\theta = \sin\theta/\cos\theta$

$\sin\theta = \sin(\theta \pm 2\pi n)$ $\quad$ $\cos\theta = \cos(\theta \pm 2\pi n)$ $\quad$ $\tan\theta = \tan(\theta \pm \pi n)$

$\cos\theta = \cos(-\theta)$ $\quad$ $\sin\theta = -\sin(-\theta)$ $\quad$ $\tan\theta = -\tan(-\theta)$

$$\sin\theta = \cos(\theta - \pi/2) = \cos(\pi/2 - \theta) = \sin(\pi - \theta)$$
$$\cos\theta = \sin(\theta + \pi/2) = \sin(\pi/2 - \theta) = -\cos(\pi - \theta)$$
$$\tan\theta = \cot(\pi/2 - \theta) = -\tan(\pi - \theta)$$

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$ $\quad$ $1 + \cot^2\theta = \csc^2\theta$

Functions of Sums and Differences of Angles

$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$ $\quad$ $\sin(\theta - \phi) = \sin\theta \cos\phi - \cos\theta \sin\phi$

$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$ $\quad$ $\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \tan\phi} \qquad \tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta \tan\phi}$$

Double-Angle Formulas

$\sin 2\theta = 2\sin\theta \cos\theta$ $\quad$ $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half-Angle Formulas

$$\sin \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 - \cos\theta}{2}} \qquad \cos \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

$$\tan \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$$

B.2.4. Inverse Trigonometric Functions

Each of the trigonometric functions has a corresponding inverse function. The common convention is to name the inverse functions with the prefix arc-: arcsine, arccosine, etc. The trig-function abbreviation is given a superscript –1 to denote the corresponding inverse function, e.g. $\cos^{-1}$ is the function abbreviation for arccosine. The inverse functions are multivalued because the trigonometric functions are periodic. For example, the arccosine of 0.5 has the values

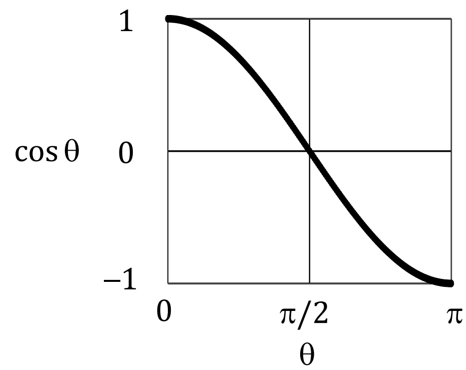
$$\cos^{-1}(0.5) = \pi/3 + 2\pi n \quad \text{and} \quad -\pi/3 + 2\pi n \qquad \text{for all integer n}$$

because

$$\cos(\pi/3 + 2\pi n) = \cos(-\pi/3 + 2\pi n) = 0.5 \qquad \text{for all integer n.}$$

The *principal* inverse trigonometric functions provide single-valued inverse functions, which are denoted by capitalizing the initial letter of the inverse function name and abbreviation. For example, the principal arccosine is written Arccosine and has the functional abbreviation Cos^{-1}. The following paragraphs describe the principal inverse of the cosine, of the sine, and of the tangent.

The <u>principal arccosine function</u> is the inverse of the cosine function over the restricted cosine domain $0 \le \theta \le \pi$. Over this domain, the function $\cos\theta$ decreases in strictly monotonic fashion from its maximum value of 1 at $\theta = 0$ to its minimum value of -1 at $\theta = \pi$. Thus, there is a unique value of θ in the region $[0,\pi]$ for each value of $\cos\theta$ where $-1 \le \cos\theta \le 1$. The principal arccosine function, denoted Cos^{-1}, is therefore defined as follows.



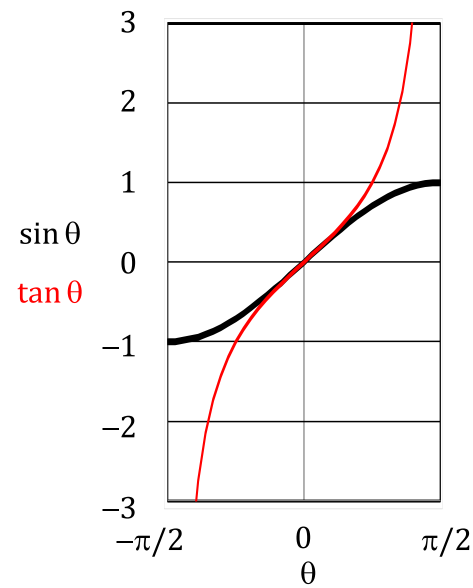
Plot of $\cos\theta$ for $0 \le \theta \le \pi$

$\theta = \mathrm{Cos}^{-1}(u)$ is the value of θ on the interval $[0,\pi]$ such that $u = \cos\theta$.

The domain of $\mathrm{Cos}^{-1}(u)$ is $-1 \le u \le 1$; the range is $0 \le \mathrm{Cos}^{-1}(u) \le \pi$.

The <u>principal arcsine function</u> is the inverse of the sine function over the restricted sine domain $-\pi/2 \le \theta \le \pi/2$. Over this domain, the function $\sin\theta$ increases in strictly monotonic fashion from its minimum value of -1 at $\theta = -\pi/2$ to its maximum value at $\theta = \pi/2$. Thus, there is a unique value of θ in the region $[-\pi/2, \pi/2]$ for each value of $\sin\theta$ where $-1 \le \sin\theta \le 1$.

The <u>principal arctangent function</u> is the inverse of the tangent function over the restricted tangent domain $-\pi/2 < \theta < \pi/2$. Over this domain, the function $\tan\theta$ increases in strictly monotonic fashion ranging over all real values of θ. Thus, there is a unique value of θ in the region $(-\pi/2, \pi/2)$ for each real value of $\tan\theta$.



Plot of $\sin\theta$ and $\tan\theta$ for $0 \le \theta \le \pi$

The principal arcsine function Sin^{-1} and the principal arctangent function Tan^{-1} are defined as follows.

$\theta = \mathrm{Sin}^{-1}(u)$ is the value of θ on the interval $[-\pi/2, \pi/2]$ such that $u = \sin\theta$.

The domain of $\mathrm{Sin}^{-1}(u)$ is $-1 \le u \le 1$; the range is $-\pi/2 \le \mathrm{Sin}^{-1}(u) \le \pi/2$.

$\theta = \mathrm{Tan}^{-1}(u)$ is the value of θ on the interval $(-\pi/2, \pi/2)$ such that $u = \tan\theta$.

The domain of $\mathrm{Tan}^{-1}(u)$ is all real values of u; the range is $-\pi/2 < \mathrm{Tan}^{-1}(u) < \pi/2$.

B.3. Properties of Complex Numbers

The following paragraphs provide a review of some properties of complex numbers including roots of real and complex numbers. A *complex number* z has the form $z = x + iy$ where x and y are each real and the *imaginary unit* i is defined as $i = \sqrt{-1}$, that is $i^2 = -1$. Number x is called the *real part* of z, and y is called the *imaginary part* of z.

B.3.1. Elementary Properties

Property 1, Equality: Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal to each other if and only if $x_1 = x_2$ and $y_1 = y_2$.

Property 2, Zero: A complex number $z = x + iy$ equals zero if and only if $x = 0$ and $y = 0$.

Property 3, Addition: The sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2).$$

Property 4, Subtraction: The difference of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is

$$z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2).$$

Property 5, Multiplication: The product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1).$$

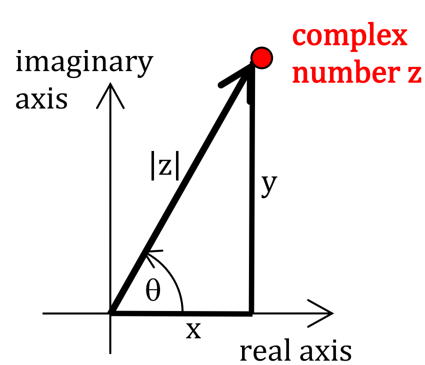
B.3.2. Polar Representation

There are two common methods of defining a complex number z. The first defines z in terms of its real part x and imaginary part y: $z = x + iy$. The second defines z in terms of its absolute value or modulus $|z|$ and its *argument* θ.

$$z = x + iy = |z|\,(\cos\theta + i\sin\theta)$$

The absolute value or modulus is computed from x and y via

$$|z| = \sqrt{x^2 + y^2}.$$



Argument θ is the angle in radian measure whose cosine is $x/|z|$ and whose sine is $y/|z|$. That is,

$$x = |z|\cos\theta \qquad y = |z|\sin\theta.$$

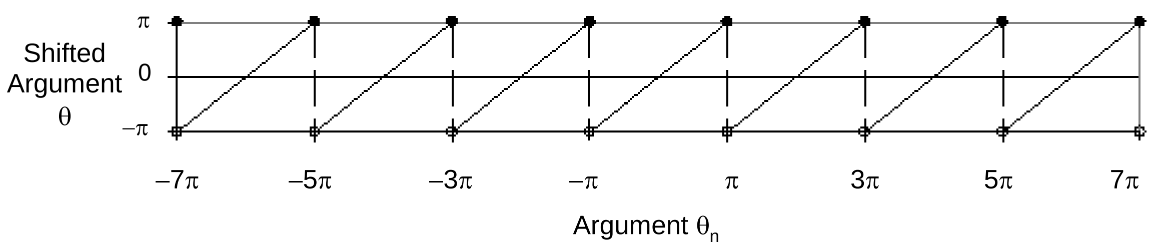
The principal value of the argument lies in the range $-\pi < \theta \le \pi$. This principal range is adequate to represent the argument of all nonzero complex numbers. Argument θ is undefined for $z = 0$. Unless specified otherwise, the argument is usually assumed to be its principal value.

The principal value of θ may be calculated from x, y, and $|z|$ using an expression containing one of the principal inverse trigonometric functions. Here we use the principal arccosine function Cos^{-1} whose domain is $[-1, 1]$ and whose range is $[0, \pi]$.

$$\theta = \begin{cases} \mathrm{Cos}^{-1}(x/|z|), & y \geq 0 \\ -\mathrm{Cos}^{-1}(x/|z|), & y < 0 \\ \text{undefined}, & x = y = 0 \end{cases}$$

B.3.3. Converting Any Argument to its Principal Value

If θ is replaced with $\theta + 2\pi n$ where n is any integer, then the value of z remains unchanged. If an argument θ_n lies outside the principal range, then the corresponding argument θ within the principal range is found by subtracting the appropriate multiple n of 2π. Specifically, $\theta = \theta_n - 2\pi n$ where n is the smallest integer that is greater than or equal to $\left(\frac{\theta_n}{2\pi} - \frac{1}{2}\right)$.



B.3.4. Euler's Formula

Euler's formula is $e^{i\theta} = \cos\theta + i\sin\theta$.

Because $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, the complex number $e^{-i\theta}$ is

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Euler's formula shows that the polar form of a complex number z may be written

$$z = |z|e^{i\theta}.$$

Conversely, $\cos\theta$ and $\sin\theta$ may be expressed as linear combinations of $e^{i\theta}$ and $e^{-i\theta}$.

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

B.3.5. Multiplication and Division in Polar Form

The product of two complex numbers $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$ is given by

$$z_1 z_2 = |z_1||z_2|e^{i(\theta_1+\theta_2)}.$$

Thus, to multiply, we find the product of moduli and the sum of arguments. Similarly, the quotient is

$$z_1/z_2 = z_1(z_2)^{-1} = \frac{|z_1|}{|z_2|} e^{i(\theta_1-\theta_2)}.$$

B.3.6. Integer Powers

Using the previous rule for multiplication, the integer n power of complex number $z = |z|e^{i\theta}$ is calculated as

$$z^n = \left(|z|e^{i\theta}\right)^n = \underbrace{\left(|z|e^{i\theta}\right)\left(|z|e^{i\theta}\right)\ldots\left(|z|e^{i\theta}\right)}_{n} = |z|^n\left(e^{i\theta}\right)^n = |z|^n e^{in\theta}.$$

Thus, to take the n^{th} power of z, we find the n^{th} power of the modulus and the n^{th} multiple of the argument. The resulting argument $n\theta$ is not necessarily a principal value. If not, it may be converted to its principal value as described in Section B.3.3 above.

B.3.7. Integer Roots

Suppose now we want the n^{th} root of $z = |z|e^{i\theta}$ where $z \neq 0$, θ is the principal argument of z, and n is an integer that is greater than or equal to 2. Let $z_1 = |z_1|e^{i\theta_1}$ be a root such that

$$z_1 = |z_1|e^{i\theta_1} = z^{1/n} = \left(|z|e^{i\theta}\right)^{1/n} = |z|^{1/n}\left(e^{i\theta}\right)^{1/n} = \sqrt[n]{|z|}\, e^{i\theta/n}.$$

The modulus of root z_1 is $|z_1| = \sqrt[n]{|z|}$, and the argument is $\theta_1 = \theta/n$.

Just as every nonzero real number has two discrete square roots, every nonzero complex number has n discrete n^{th} roots. Root z_1 is only one of n discrete roots of z. Recall that adding an integer multiple of 2π to the argument does not change the value of the complex number. Thus, our original complex number z may be expressed

$$z = |z|e^{i\theta} = |z|e^{i[\theta+(k-1)2\pi]}$$

for any integer k. Using this expression for z, we may generalize our derivation of z_1 to that of all discrete roots z_k, $1 \leq k \leq n$.

$$z_k = z^{1/n} = \left(|z|e^{i[\theta+(k-1)2\pi}\right)^{1/n} = |z|^{1/n}\, e^{i[\theta/n+2\pi(k-1)/n]}$$

$$\boxed{z_k = \sqrt[n]{|z|}\, e^{i[\theta/n+2\pi(k-1)/n]}, \quad k = 1, 2, \ldots, n}$$

We see that the arguments of the various roots differ from each other by multiples of $2\pi/n$. If root z_k has a calculated argument that is not a principal value, then the argument may be converted to its principal value as described in Section B.3.3 above.

<u>Example: Cube Root of –8</u>

As an example, let us find the three cube roots of $z = x + iy = -8$. Then $x = -8$, $y = 0$, and $n = 3$. The modulus of z is

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(-8)^2 + 0^2} = \sqrt{64} = 8.$$

Section B.3.2 shows that since $y \geq 0$, θ may be calculated as

$$\theta = \text{Cos}^{-1}(x/|z|) = \text{Cos}^{-1}(-8/8) = \text{Cos}^{-1}(-1) = \pi.$$

Substituting $n = 3$, $|z| = 8$, and $\theta = \pi$ into the formula for the roots z_k produces

$$z_k = \sqrt[3]{8}\, e^{i[\pi/3+2\pi(k-1)/3]} = 2\, e^{i[\pi/3+2\pi(k-1)/3]}, \quad k = 1, 2, 3.$$

The three roots z_k for $k = 1, 2, 3$ are then:

$z_1 =$	$2\,e^{i\pi/3} = 2[\cos(\pi/3) + i\sin(\pi/3)] = 2[1/2 + i\sqrt{3}/2] =$	$1 + i\sqrt{3}$
$z_2 =$	$2\,e^{i(\pi/3+2\pi/3)} = 2e^{i\pi} = 2[\cos(\pi) + i\sin(\pi)] = 2[-1 + i0] =$	-2
$z_3 =$	$2e^{i(\pi/3+4\pi/3)} = 2e^{i(5\pi/3)} = 2e^{i(5\pi/3-2\pi)} = 2e^{i(-\pi/3)} =$ $2[\cos(-\pi/3) + i\sin(-\pi/3)] = 2[1/2 - i\sqrt{3}/2] =$	$1 - i\sqrt{3}$

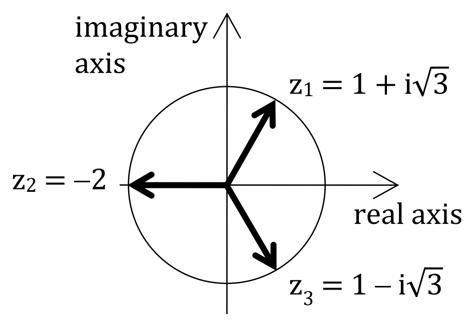
The three cube roots of $z = -8$ are: $1 + i\sqrt{3}$, -2, $1 - i\sqrt{3}$.

To verify each root, check that its cube is equal to -8.

$(1 + i\sqrt{3})^3 = 1^3 + 3(1)^2 i\sqrt{3} + 3(1)(i\sqrt{3})^2 + (i\sqrt{3})^3 = 1 + i3\sqrt{3} + 3(-3) - i3\sqrt{3} = -8$ ✓

$(-2)^3 = -8$ ✓

$(1 - i\sqrt{3})^3 = 1^3 + 3(1)^2(-i\sqrt{3}) + 3(1)(-i\sqrt{3})^2 + (-i\sqrt{3})^3 = 1 - i3\sqrt{3} + 3(-3) + i3\sqrt{3} = -8$ ✓



Cube Roots z_1, z_2, z_3 of -8

B.3.8. Complex Conjugate Properties

<u>Definition:</u> The complex conjugate of $z = x + iy$ is denoted z^* and is equal to $z^* = x - iy$. A complex number z and its conjugate z^* have real parts that are equal to each other and imaginary parts of equal magnitude and opposite sign.

<u>Symmetry</u>: If z^* is the complex conjugate of z, then z is the complex conjugate of z^*. That is, $z^{**} = z$.

$$z^{**} = [(x + iy)^*]^* = (x - iy)^* = x + iy = z$$

Polar Representation: A complex number z and its conjugate z* have moduli that are equal to each other and arguments of equal magnitude and opposite sign.

$$z = x + iy = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$$

$$z^* = x - iy = |z|(\cos\theta - i\sin\theta) = |z|[\cos(-\theta) + i\sin(-\theta)] = |z|e^{-i\theta}$$

Integer Powers and Roots: For integer n, the n^{th} power of z* is the complex conjugate of z^n.

$$(z^*)^n = \left(|z|e^{-i\theta}\right)^n = |z|^n e^{-in\theta} = (z^n)^*$$

Similarly, if $z^{1/n}$ is any n^{th} root of z, then $(z^{1/n})^*$ is an n^{th} root of z*.