Selected Algorithms

Each of the first two pages shows an algorithm for solving the cubic equation and one for solving the quartic algorithm. Calculated solutions may be checked as shown on page 3. When coding the cubic-equation algorithms, interpret the cube-root operation $y = x^{1/3}$ as:

$$y = x^{1/3} \text{ for } x \geq 0; \quad y = -(-x)^{1/3} \text{ for } x < 0.$$  

Cardano-Viète Algorithm for Solving the Cubic Equation

Note: All angle values are in radian measure (180 degrees $= \pi$ radians).

<table>
<thead>
<tr>
<th>Given:</th>
<th>Real coefficients $a_2$, $a_1$, and $a_0$,</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find:</td>
<td>$z_1, z_2=x_2+iy_2$, and $z_3=x_3+iy_3$ such that $z^3 + a_2 z^2 + a_1 z + a_0 = (z-z_1) (z-z_2) (z-z_3)$ for all $z$.</td>
</tr>
<tr>
<td>Calculate $q$ and $r$:</td>
<td>$q = \frac{a_1}{3} - \frac{a_2^2}{9}$ \quad $r = \frac{a_1 a_2 - 3a_0}{27}$</td>
</tr>
</tbody>
</table>

Case 1: $r^2 + q^3 > 0 \iff$ Only One Real Solution (Cardano)

- $u = (r + \sqrt{r^2 + q^3})^{1/3}$
- $v = (r - \sqrt{r^2 + q^3})^{1/3}$
- $z_1 = u + v - a_2/3$
- $x_2 = x_3 = -\frac{u+v}{2} - \frac{a_2}{3}$
- $y_2 = -y_3 = \frac{-\sqrt{3}(u-v)}{2}$

Note: $u$ and $v$ are real cube roots of real numbers. Also, $u > v$.

Case 2: $r^2 + q^3 \leq 0 \iff$ Three Real Solutions (Viète)

$$\theta = \begin{cases} 0 & \text{if } q = 0 \\ \frac{1}{3} \cos^{-1}\left(\frac{r}{\sqrt{-q}}\right) & \text{if } q < 0 \end{cases}$$

- $\phi_1 = \theta/3$
- $\phi_2 = \phi_1 - 2\pi/3$
- $\phi_3 = \phi_1 + 2\pi/3$
- $z_1 = 2\sqrt{-q} \cos \phi_1 - a_2/3$
- $z_2 = x_2 = 2\sqrt{-q} \cos \phi_2 - a_2/3$
- $z_3 = x_3 = 2\sqrt{-q} \cos \phi_3 - a_2/3$

Note: $z_3 \leq z_2 \leq z_1$

Note: For the special case $0 < |q^3| < r^2$, a typical 64-bit operating system using this Cardano-Viète Algorithm suffers increased solution error of up to 5 parts per million. The longer All-Trigonometric Algorithm on the following page does not have such a problem. See Details.

National Bureau of Standards (NBS) Modified Algorithm for Solving the Quartic Equation

<table>
<thead>
<tr>
<th>Problem:</th>
<th>Given real coefficients $A_3, A_2, A_1$, and $A_0$, find $Z_1, Z_2, Z_3$ and $Z_4$ such that $Z^4 + A_3 Z^3 + A_2 Z^2 + A_1 Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4)$ for all $Z$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution:</td>
<td>Use either cubic-equation algorithm to calculate $u_1$ as the greatest real solution of the resolvent cubic equation $u^3 - A_2 u^2 + (A_1 A_3 - 4A_0) u + 4A_0 A_2 - A_1^2 - A_0 A_3^2 = 0$.</td>
</tr>
</tbody>
</table>

$$\Sigma_e = \begin{cases} 1 & \text{if } A_1 - A_3 u_1/2 > 0 \\ -1 & \text{otherwise} \end{cases}$$

- $p_1 = A_3/2 - \sqrt{A_3^2/4 + u_1 - A_2}$
- $p_2 = A_3/2 + \sqrt{A_3^2/4 + u_1 - A_2}$
- $q_1 = u_1/2 + \Sigma_e \sqrt{u_1^2/4 - A_0}$
- $q_2 = u_1/2 - \Sigma_e \sqrt{u_1^2/4 - A_0}$
- $Z_{1,2} = -p_1/2 \pm \sqrt{p_1^2/4 - q_1}$
- $Z_{3,4} = -p_2/2 \pm \sqrt{p_2^2/4 - q_2}$

Note: Wolters’ modifications in red allow the algorithm to convert easily to code. Using $u_1$ as the greatest real solution of the resolvent cubic equation assures that calculated values $p_1, p_2, q_1$, and $q_2$ are all real numbers. The value $\Sigma_e$ assures that the formulas for $q_1$ and $q_2$ use the correct sign for the radical term.
All-Trigonometric Algorithm for Solving the Cubic Equation

Note: All angle values are in radian measure (180 degrees = π radians).

Given: Real coefficients \(a_2, a_1,\) and \(a_0,\)

Find: \(z_1, z_2=x_2+iy_2,\) and \(z_3=x_3+iy_3\) such that
\[z^3 + a_2 z^2 + a_1 z + a_0 = (z-z_1)(z-z_2)(z-z_3)\] for all \(z.\)

Calculate \(q\) and \(r:\)
\[q = \frac{a_1}{3} - \frac{a_2^2}{9}\]
\[r = \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27}\]

Case A: \(r^2 + q^3 > 0,\) \(q < 0 \Rightarrow\) Only One Real Solution
\[\gamma = \sin^{-1}[(-q)^{3/2} / r],\quad 0 < |\gamma| < \pi/2\]
\[\chi = \tan^{-1}[[\tan(\gamma/2)]^{1/3}],\quad 0 < |\chi| < \pi/4\]
\[z_1 = 2\sqrt{-q} \csc 2\chi - a_2/3\]
\[z_2 = x_2 + i y_2\]
\[z_3 = x_2 - i y_2\]

Case B: \(r^2 + q^3 > 0,\) \(q = 0 \Rightarrow\) Only One Real Solution
\[z_1 = (2r)^{1/3} - a_2/3\]
\[z_2 = x_2 + i y_2\]
\[z_3 = x_2 - i y_2\]

Case C: \(r^2 + q^3 > 0,\) \(q > 0 \Rightarrow\) Only One Real Solution
\[\gamma = \{\pi/2\} \begin{cases} \pi/2 & \text{if } r = 0, \\ \tan^{-1}(q^{1/2}/r) & \text{if } r \neq 0, \end{cases},\quad 0 < |\gamma| \leq \pi/2\]
\[\chi = \tan^{-1}[[\tan(\gamma/2)]^{1/3}],\quad 0 < |\chi| \leq \pi/4\]
\[z_1 = 2\sqrt{q} \cot 2\chi - a_2/3\]
\[z_2 = x_2 + i y_2\]
\[z_3 = x_2 - i y_2\]

Case D: \(r^2 + q^3 \leq 0 \Leftrightarrow\) Three Real Solutions (Viète)
\[\theta = \begin{cases} 0 & \text{if } q = 0, \\ \cos^{-1}(\max\{-1, \min[r/(-q)^{3/2}, 1]\}) & \text{if } q < 0 \end{cases}\]
\[0 \leq \theta \leq \pi\]
\[\phi_1 = 0/3\]
\[\phi_2 = \phi_1 - 2\pi/3\]
\[\phi_3 = \phi_1 + 2\pi/3\]
\[z_1 = 2\sqrt{-q} \cos \phi_1 - a_2/3\]
\[z_2 = x_2 + i y_2\]
\[z_3 = x_2 - i y_2\]

Ferrari Modified Algorithm for Solving the Quartic Equation

Given: Real coefficients \(A_3, A_2, A_1,\) and \(A_0,\)

Find: \(z_1, z_2, z_3, z_4\) such that
\[Z^4 + A_3 Z^3 + A_2 Z^2 + A_1 Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4)\] for all \(Z.\)

Calculation: \(C = A_3/4,\)
\[b_2 = A_2 - 6C^2,\]
\[b_1 = A_1 - 2A_2C + 8C^3,\]
\[b_0 = A_0 - A_1C + A_2C^2 - 3C^4\]

Use either cubic-equation algorithm to solve this resolvent cubic equation for real \(m:\)
\[m^3 + b_2 m^2 + (b_1^2/4 - b_0)m - b_1^2/8 = 0.\] Use a real solution \(m > 0\) if it exists. Otherwise, \(m = 0.\)

\[\Sigma = \begin{cases} 1 & \text{if } b_1 > 0, \\ -1 & \text{otherwise} \end{cases}\]
\[R = \Sigma \sqrt{m^2 + b_2 m + b_1^2/4 - b_0}\]
\[Z_{1,2} = \sqrt[3]{m/2 - C} \pm \sqrt[3]{-m/2 - b_2/2 - R}\]
\[Z_{3,4} = -\sqrt[3]{m/2 - C} \pm \sqrt[3]{-m/2 - b_2/2 + R}\]

Note: Wolters’ modifications in red assure that the algorithm is computationally stable and that the real value \(R\) is calculated with the correct sign.

6/9/2020
Validate Calculated Solutions of the Cubic Equation

Validate calculated solutions \(z_1, z_2 = x_2 + iy_2,\) and \(z_3 = x_3 + iy_3\) by reproducing the input coefficients according to the following check equations:

\[
a_2 = -(z_1 + x_2 + x_3) \quad a_1 = z_1(x_2 + x_3) + x_2x_3 + y_2^2 \quad a_0 = -z_1(x_2x_3 + y_2^2).
\]

Validate Calculated Solutions of the Quartic Equation

Validate calculated solutions \(Z_1 = X_1 + iY_1, Z_2 = X_2 - iY_1, Z_3 = X_3 + iY_3,\) and \(Z_4 = X_4 - iY_3\) by reproducing the input coefficients according to the following check equations:

\[
A_3 = -(X_1 + X_2 + X_3 + X_4)
\]
\[
A_2 = X_1X_2 + Y_2^2 + (X_1 + X_2)(X_3 + X_4) + X_3X_4 + Y_3^2
\]
\[
A_1 = -[(X_1X_2 + Y_2^2)(X_3 + X_4) + (X_3X_4 + Y_3^2)(X_1 + X_2)]
\]
\[
A_0 = (X_1X_2 + Y_2^2)(X_3X_4 + Y_3^2).
\]
Assessment of the Cubic-Equation Algorithms for General Calculation
(taken from page 3 of Practical Algorithms for Solving the Cubic Equation)

Cardano-Viète is the simpler of the two algorithms, but it produces relatively large solution error due to round-off when \( 0 < |q^3| << r^2 \). To examine this situation, consider Cardano’s formula for the depressed solution \( t_1 = z_1 + a_2/3 = u + v \) with \( r \) set equal to 1.

\[
t_1(q) = \left( 1 + \sqrt{1 + q^3} \right)^{1/3} + \left( 1 - \sqrt{1 + q^3} \right)^{1/3}
\]

We want to examine cases in which \( |q^3| << 1 \). For \( q = 0 \), we have \( t_1(0) = 2^{1/3} \). Because \( t_1(q) \) is a function of \( 1 + q^3 \), a computer using Cardano cannot calculate a \( t_1 \) deviation from \( t_1(0) \) unless \( |q^3| \) exceeds the computer’s precision limit. This limit is on the order of \( 10^{-16} \) for the 64-bit operating system we use below. For \( |q^3| \) to exceed \( 10^{-16} \), \( |q| \) must exceed \( 10^{-16/3} \) or about \( 10^{-6} \) to \( 10^{-5} \). For smaller \( |q| \) values, calculated \( t_1(q) \) is stuck at \( t_1(0) = 2^{1/3} \).

The figure demonstrates this situation graphically. It shows that using Cardano, the computer cannot detect \( t_1(q) \) deviations from \( t_1(0) \) for small values of \( q \). By contrast, the All-Trigonometric Algorithm produces accurate results for small \( q \) as expressed below:

\[
t_1(q) - 2^{1/3} \approx \left[ \frac{dt_1(q)}{dq} \right]_{q=0} q = -2^{-1/3} q \text{ for } |q| << 1.
\]

Note: The appendix in Practical Algorithms for Solving the Cubic Equation derives the derivative value

\[
\left. \frac{dt_1}{dq} \right|_{q=0} = -\frac{1}{t_1(0)} = -2^{-1/3}.
\]

Return to Selected Algorithms