SELECTED ALGORITHMS

PRACTICAL ALGORITHM FOR SOLVING THE CUBIC EQUATION

Given: Real coefficients a_2, a_1, and a_0,

Find: z_1, $z_2=x_2+iy_2$, and $z_3=x_3+iy_3$ such that $z^3 + a_2 z^2 + a_1 z + a_0 = (z - z_1)(z - z_2)(z - z_3)$ for all z.

Outputs z_1, z_2, and z_3 are solutions of the cubic equation $z_n^3 + a_2 z_n^2 + a_1 z_n + a_0 = 0$, $n = 1, 2, 3$.
Solution z_1 is the greatest real solution.

Calculate q and r: $q = \frac{a_1}{3} - \frac{a_2^2}{9}$ $\quad r = \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27}$

Case: 1: $r^2 + q^3 > 0$ ⇔ Only One Real Solution
(*Numerical Recipes §5.6 **)

$$A = \left(|r| + \sqrt{r^2 + q^3}\right)^{1/3}$$

$$t_1 = \begin{cases} A - q/A & \text{if } r \geq 0 \\ q/A - A & \text{if } r < 0 \end{cases}$$

$$z_1 = t_1 - \frac{a_2}{3} \qquad x_2 = x_3 = -\frac{t_1}{2} - \frac{a_2}{3}$$

$$y_2 = -y_3 = \frac{\sqrt{3}}{2}\left(A + \frac{q}{A}\right)$$

$$z_2 = x_2 + iy_2 \qquad z_3 = x_2 - iy_2$$

Case 2: $r^2 + q^3 \leq 0$ ⇔ Three Real Solutions
(Viète)

$$\theta = \begin{cases} 0 & \text{if } q = 0 \\ \mathrm{Cos}^{-1}[r/(-q)^{3/2}] & \text{if } q < 0 \end{cases} \qquad 0 \leq \theta \leq \pi$$

$$\phi_1 = \theta/3 \qquad \phi_2 = \phi_1 - 2\pi/3 \qquad \phi_3 = \phi_1 + 2\pi/3$$

$$z_1 = 2\sqrt{-q}\,\cos\phi_1 - a_2/3$$

$$z_2 = x_2 = 2\sqrt{-q}\,\cos\phi_2 - a_2/3 \qquad y_2 = 0$$

$$z_3 = x_3 = 2\sqrt{-q}\,\cos\phi_3 - a_2/3 \qquad y_3 = 0$$

$$z_3 \leq z_2 \leq z_1$$

* Press, W.H., et al., *Numerical Recipes. The Art of Scientific Computing*, 3rd Edition, 2007, Cambridge University Press, ISBN 978-0-521-88068-8, http://numerical.recipes/book/book.html.

MODIFIED EULER ALGORITHM FOR SOLVING THE QUARTIC EQUATION

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that $Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4)$ for all Z.

The outputs are thus the four solutions of the general quartic equation

$$Z_n^4 + A_3 Z_n^3 + A_2 Z_n^2 + A_1 Z_n + A_0 = 0, n = 1, 2, 3, 4.$$

Calculation: $C = A_3 / 4$, $\quad b_2 = A_2 - 6C^2$, $\quad b_1 = A_1 - 2A_2C + 8C^3$, $\quad b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

$$\Sigma = 1 \text{ if } b_1 > 0,\ \Sigma = -1 \text{ otherwise.}$$

Find the three solutions r_1, r_2, and r_3 of the resolvent cubic equation:

$$r_k^3 + (b_2/2)\, r_k^2 + [(b_2^2 - 4b_0)/16]\, r_k - b_1^2 / 64 = 0.$$

Solution r_1 is the greatest real solution and $r_1 \geq 0$. Solutions $r_2 = x_2 + iy_2$ and $r_3 = x_3 + iy_3$ are real ($y_2 = y_3 = 0$), or they form a complex conjugate pair ($x_2 = x_3$, $y_2 = -y_3 > 0$).

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{x_2 + x_3 - 2\Sigma\sqrt{x_2x_3 + y_2^2}} \qquad T_{3,4} = -\sqrt{r_1} \pm \sqrt{x_2 + x_3 + 2\Sigma\sqrt{x_2x_3 + y_2^2}}$$

where $x_2x_3 + y_2^2 \geq 0$.

$$Z_n = T_n - C, \quad n = 1, 2, 3, 4$$

Note: Wolters' modifications in red allow the algorithm to be executed using operations on real numbers only.

VALIDATE CALCULATED SOLUTIONS OF THE CUBIC EQUATION

Validate calculated solutions z_1, $z_2 = x_2 + iy_2$, and $z_3 = x_3 + iy_3$ by reproducing the input coefficients according to the following check equations:

$a_2 = -(z_1 + x_2 + x_3)$ $\quad a_1 = z_1(x_2+x_3) + x_2x_3 + y_2^2$ $\quad a_0 = -z_1(x_2x_3+y_2^2)$.

VALIDATE CALCULATED SOLUTIONS OF THE QUARTIC EQUATION

Validate calculated solutions $Z_1 = X_1 + iY_1$, $Z_2 = X_2 - iY_1$, $Z_3 = X_3 + iY_3$, and $Z_4 = X_4 - iY_3$ by reproducing the input coefficients according to the following check equations:

$$A_3 = -(X_1+X_2+X_3+X_4)$$

$$A_2 = X_1X_2+Y_1^2+(X_1+X_2)(X_3+X_4)+X_3X_4+Y_3^2$$

$$A_1 = -[(X_1X_2+Y_1^2)(X_3+X_4) + (X_3X_4+Y_3^2)(X_1+X_2)]$$

$$A_0 = (X_1X_2+Y_1^2)(X_3X_4+Y_3^2).$$